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APPLIED DIFFERENTIAL EQUATIONS

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PREFACE

The outstanding characteristic of modern mathematics is the rigour that has come into analysis. The ideas of a mathematician, self-taught from the works of Todhunter, would need considerable reorientation in the presence of a modern text on, say, the calculus of variations. The like cannot be said of the worker in applied science, and the man who learns his mathematics from John Perry manages to get along quite well.

This is just as it should be. "Epsilonology" is to the mathematician what J. S. Bach is to the musician; there is an enormous sense of gratification in its essential rightness. It is an exhilarating experience to go right down to the foundations of one's subject; especially if one does not stay there. In later years, most of us mathematicians recapture some of this charm by renewed visits. But I see no justification on that account for expecting those to whose studies mathematics is ancillary to master the philosophical profundities of a limit. It would be as rational to expect the piano-tuner to be interested in the comma of Pythagoras. Geometrical intuition is good enough for the applied scientist. Anyway, he can always salve his conscience at a later date by undertaking a critical survey. Actually he would be better employed in getting on with his job, especially if he remembers that the professional mathematician puts in years of advanced study before he can (if he ever does) establish even the binomial theorem in all its generality.

Shortly after this book was first set up in type I noticed signs in various quarters of a move to reform the teaching of mathematics to science students. Having twice in my life had the illuminating experience of being a professional mathematician in a large research station I welcome the move. Most of the trouble is that both teachers and examiners are mathematicians solely. I have often sympathized with the late G. H. Bryan, who complained that his assistants could give him answers in terms of "cosh minus one" but could not put them into figures. I have been more than once faintly amused (and mildly annoyed) to receive a new assistant, the laurels of his honours degree still fresh on his brow, and to see that brow furrow when faced with the task of working some examples for a class.

Here then is the book. I wrote it for two excellent reasons: firstly that I thought it needed writing, and secondly that I thought I could make a reasonably good job of it. It is characterized by a frankness of statement unusual in textbooks. In one or two places there is a faint odour of Lie's theory of continuous groups (though actually I got it from Page); but the uninitiated will not notice this, and the *cognoscenti* will not find it pronounced enough to be offensive.

For the rest, its title describes it. The standard methods of solution are illustrated by such applications from other subjects as come in one's student years. The perfect differential and the integration of the exact equation are given generous treatment, for the reasons given in the text. The chapter on numerical work is not usually included, and the chapter on isoclinals rarely appears at all. My one regret here is limiting myself to a mere reference to the work of van der Pol, which should receive adequate treatment in any text specifically devoted to the subject.

The examples number rather more than four hundred. This could easily have been doubled by the inclusion of academic exercises, to the detriment of the book. I am no believer in problem-grinding, and anyway the examiner can always pose to the candidate an unfamiliar example that is neither unreasonably difficult nor obvious; that is what examiners are for. Instead, there is an unusually high proportion of problems taken from scientific papers and researches. In most cases the answers are juxtaposed; in other cases the solution, or a hint at the method of approach, is given at the back of the book. In a few cases no hint or answer is supplied. This was done deliberately, on the grounds that ultimately the reader must learn to stand on his own feet and trust his own judgment.

Writing a book is not a task to be envisaged lightly, and I am confident that this book would not have been written except for two reasons. Some years ago I had the good fortune to be in the ideal *milieu*, on the staff of the Imperial College of Science and Technology, and it is perfectly transparent (to me, at least) that this exposition is more than tinged with the thought and methods of my colleagues there. In particular, I was under the ægis of Prof. H. Levy, and nobody could come within the orbit of his fertile mind without acquiring a beneficent rise of potential. I am indebted also to the German engineer, Wilhelm Hort, the appendix to whose book proved a mine of references to original papers.

WATFORD,
April, 1947.

F. E. RELTON.

CONTENTS

The paragraphs are numbered consecutively throughout a chapter. The equations are numbered consecutively throughout a paragraph. A reference such as 2, 8 (iv) means chapter 2, paragraph 8, equation (iv) therein. A decimal point such as 3, 4·1 denotes a paragraph subsidiary to 3, 4. These can usually be omitted on a first reading.

CHAPTER I

Preliminary Notions; Hyperbolic Functions

1, 1. The best way of beginning the practice of differential equations is to see one solved. We suppose, then, that in the perusal of a scientific journal your eye lights on the following,

$$\frac{dy}{dx} = \frac{1 - 2x}{y^2}.$$

It obviously implies a relation between two variables x and y, and states categorically that the right-hand side is equal to the differential coefficient of y with respect to x. It is a differential equation, and though direct integration is out of the question, there seems no reason why we should not write it in the form

$$y^2\,dy = (1 - 2x)\,dx,$$

which leads to $\qquad \int y^2\,dy = \int (1 - 2x)\,dx + \text{constant},$

whence we derive $\qquad \tfrac{1}{3}y^3 = x - x^2 + c.$

That seems about as far as we can get until you notice, a few lines farther down in your hypothetical journal, one more intelligible remark which reads $x = 1$, $y = 0$. The substitution of these values decides that c is zero, leaving the relation $y^3 = 3x(1 - x)$.

You have now solved a first-order non-linear ordinary differential equation by a method known as the separation of the variables, and you have determined the arbitrary constant from assigned conditions. A good deal of this last statement is probably unintelligible at the moment, but that is immaterial, since we are merely taking a first glance and shall come back to the details later.

1, 1·1. Part of the foregoing flies in the face of one's earliest lessons in the calculus. It is there impressed upon one that the dy and dx in a differential coefficient have lost their individuality and become merged in a limiting ratio. The justification for restoring their individuality by tearing the differential coefficient apart and placing dy and dx on opposite sides of the equality sign, thus replacing a differential coefficient by two differentials, can be found in *A Course of Pure Mathematics*, by G. H. Hardy. This is easily the best book of its kind.

1, 2. The rest of this chapter calls for no intensive study, and it can be perused in leisurely fashion. Most of the ideas will be already familiar, and are inserted to refresh the memory. Any new concepts that are introduced are easily assimilable and will be encountered repeatedly in the later parts of the book.

We begin with a few fundamental ideas in the calculus. If two numbers are so related that the one is determinate whenever the other is known, the one is said to be a function of the other. This is not a strict mathematical definition of functionality; but for working purposes it will suffice. Such a relation is usually written $y = f(x)$; or $\phi(x, y) = 0$. In the former, y is said to be an explicit function of x, and it is usually understood that the relation is reversible, so that if need be, it could be considered with x as a function of y, or $x = F(y)$. For example, if an ellipse has an axis of 5″, then the other axis c is determinate if the perimeter s is known. Conversely, the perimeter s is determinate if the other axis c is known, and the one is a function of the other; $s = f(c)$ and $c = F(s)$. The fact that the relationship is not expressible in a simply calculable form is irrelevant to the idea of functionality.

In the second form $\phi(x, y) = 0$, the one variable is said to be an implicit function of the other, since the direct dependence of the one on the other is more implied than overtly stated. It is then usually a matter of choice or indifference which variable is regarded as a function of the other.

A distinction is made between the two variables. The one which renders the calculation of the other easier is usually denoted by x, and is called the independent variable. The second variable is called y and is calculable from x, or is a function of x. Thus, if the relation were

$$y = \log x + \cos(1 - x)$$

it is obviously easy to calculate y for a given x; it would be far more difficult to calculate x for a given y, and it would be insuperably difficult to express x as an explicit function of y. If the relation were of the form

$$2x^2 + 3y^2 - 2x - 5y = 17,$$

it would be at one's pleasure to decide which variable was to be regarded as independent.

It may well happen that the calculation supplies more than one answer. Even in so simple a relation as $y = \sin^{-1}x$, the value of y corresponding to $x = \frac{1}{2}$ may be $\pi/6$ or $5\pi/6$, and to each of these values any positive or negative integral multiple of 2π may be added. It is assumed in such cases that we are in possession of some criterion which enables us to select the required value; otherwise expressed, y is regarded as a one-valued function of x.

1, 3. *Notation.*

It almost invariably happens when y is a function of x that y possesses a differential coefficient with respect to x. The exceptions are the playthings of the mathematicians and need not detain us. This differential coefficient is variously written $\frac{dy}{dx}$, dy/dx or y', where the second form is called "solidus notation", and in the last one the little mark at the top is known to the printer as a "prime", and is nearly always spoken of as a dash. This differential coefficient is a function of x, and in turn has a differential coefficient, known as the second differential coefficient of y with respect to x, which is written $\frac{d^2y}{dx^2}$, d^2y/dx^2 or y''; and so on. The prime notation is useful and succinct when the independent variable is not in doubt.

A common exception is when the independent variable is the time t. The first and second differential coefficients of x with respect to t are then, especially in dynamics, frequently written $\dot{x}$ and $\ddot{x}$ respectively. The foregoing differential coefficients are known as ordinary, which distinguishes them from those which occur when more than two variables are connected by a single relation, where the resulting differential coefficients are known as partial. We shall encounter these later in the book; for the present we confine ourselves to two variables and the resulting differential coefficients which are ordinary.

1, 4. An ordinary differential equation may be defined as a relation between some of the various differential coefficients y', y'', &c., and possibly one or both of the variables x and y. As examples we have

$$\frac{dy}{dx} + y\cos x = \sin x,$$

$$\frac{d^2y}{dx^2} + \omega^2 y = 0,$$

$$(1 + y'^2)^3 = (ay'')^2.$$

To solve such an equation means to find a functional relation between the two variables, so that when the one is known the other can be calculated. The solution sometimes takes the form of having each variable expressed in terms of a third number, so that the variables are given parametrically, just as the co-ordinates of a point on an ellipse are often conveniently taken in the form $x = a\cos\theta$, $y = b\sin\theta$. It may even happen that the answer appears as an integral, and this is still considered a solution even though it cannot be evaluated in finite terms.

1, 4·1. It is of interest to note that the proportion of soluble equations is very low. Only a small fraction out of all the possible equations that could be written down would prove on examination to be tractable. This is so much the case that mathematicians have long since abandoned the attempt to find new methods of exact solution, and they now devote their attention to other aspects of the subject instead. The result is that for practical purposes one is called upon to study only a few standard types, and to be familiar with the variations and adaptations of these. The difficulties which they can provide are usually more than enough for the average student.

1, 5. Every science needs a terminology and the study of differential equations is no exception to the rule. The independent variable x plays no part here, the terminology being based solely on the dependent variable and its differential coefficients, i.e. on $y, y', y'', \ldots$.

The *order* of an equation is the order of the highest differential coefficient which is present. Reverting to the equations above, the first is of the first order; the other two are of the second order, since they both contain the second (but no higher) differential coefficient. The fact that the last of them contains y'' to the second power is irrelevant to the order.

The further specification of an equation presumes that it is written in rational, integral form. This means that there are present no fractional powers which directly affect $y, y', \ldots$, nor do any of these occur in the denominator of any fraction. Such an equation as

$$\frac{(1+y'^2)^{3/2}}{y''} = (1+x)^{1/3}$$

would be deemed to have been re-written as

$$y''^2(1+x)^{2/3} = (1+y'^2)^3,$$

where the index 2/3 is irrelevant to our specification. The *degree* of an equation is then the degree of the highest differential coefficient, so that the above equation is not only of the second order but of the second degree in that order.

An equation is said to be *linear* when only the first powers of y, y', y'', . . . are present, and there are no products of these. Otherwise expressed, the equation can be written in the form

$$a + by + cy' + dy'' + \ldots = 0,$$

where the coefficients a, b, c, . . . may be constants or any functions of x whatever. This is one of the most important forms that we shall encounter. Reverting to the three equations in 1, 4 the first two are linear and the third is non-linear.

1, 6. Differential equations occur naturally when some type of change proceeds at a variable rate in accordance with a known law. As a first example, consider the vertical free motion of a body of mass m. If its upward velocity at any instant be v, the corresponding acceleration is dv/dt or $\dot{v}$. The motion is retarded both by gravity and by atmospheric resistance. If this latter be denoted by R and happens to be proportional to some power of the velocity, we have $R = kv^n$. The laws of motion then give $m\dot{v} = -mg - kv^n$ as the non-linear first order differential equation connecting the velocity with the time.

As a second example, consider a solid being dissolved by a liquid. If the solid has an initial mass m which degenerates to x at any subsequent time t, the dissolved part of the mass is $(m - x)$. This determines the strength of the solution, which in turn influences the rate of further dissolving. If the latter is inversely proportional to the former, we get the differential equation

$$-\dot{x} = k/(m - x),$$

where k is the constant of proportionality and the minus on the left is due to the x decreasing with time.

1, 6·1. A point here arises which calls for a certain amount of notice. It is no concern of the mathematician as such how far the equation truly represents the phenomenon; that is the scientist's business to decide. The mathematician comes into action only after the scientist has made up his mind what form the equation shall take. It is only fair to point out in the former of the above examples that no such simple law of atmospheric resistance is likely to hold over a wide range of velocities, and other factors would have to be considered if the body rose to any considerable height. Temperature variation would be important for a better representation of the truth of the second example, and some account would probably have to be taken of the area of the solid-liquid interface.

1, 7. In pure mathematics the variables x and y are usually taken as cartesian co-ordinates. Differential equations then appear as the expression of some geometrical property. It is known, for example, that the radius of curvature ρ of a plane curve is given by $\rho = (1 + y'^2)^{3/2}/y''$ so that if one sought a curve whose radius of curvature at any point was proportional to the distance of that point from a fixed straight line, one would simplify the problem by taking the fixed straight line as OY and say $\rho \propto x$. This would give the differential equation

$$(1 + y'^2)^3 = k(xy'')^2.$$

A relation between x and y is usually visualized as a graph or curve. If the equation contains a literal constant or parameter c, so that $f(x, y, c) = 0$, each numerical value of c gives a separate curve, and as there is an unlimited number of values for c, we get an infinity of curves. These are known as a one-parameter family. Thus, all the straight lines which meet OX at 45° (and not 135°) are embodied in the formula $y = x + c$, where the parameter represents the intercept on OY and varies from one line to another. Similarly, if c is positive,

$$x^2 + y^2 - 2c(x + y) + c^2 = 0 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

represents all the circles that lie in the first quadrant and touch both axes of co-ordinates. The parameter c is the radius and varies from circle to circle. One-parameter families are common enough in science. The investigation under conditions of constant temperature of a relation between two variables gives a graph of the resulting observations made under isothermal conditions. The same experiment repeated at a different temperature would probably result in a slightly different graph, and we thus get a family of curves with temperature as the parameter.

Naturally it is possible to have more than one parameter. All ellipses whose principal axes are the co-ordinate axes are included in the two-parameter family $(x/a)^2 + (y/b)^2 = 1$, where a, b are the parameters. Similarly, all ellipses whose two foci are on OX are in the three-parameter family $(x + c)^2/a^2 + y^2/b^2 = 1$. The same thing applies in science where an experiment might be performed under controlled conditions of both temperature and humidity; or where a metal under test in the form of a rectangular bar could have the length, breadth and thickness all variable at will, the actual relation under investigation being between, say, sag and load.

It might be desirable in certain circumstances to consider a family as a unity apart from its parameter or parameters, to have the parameters out of the way or eliminated. The result is a property common to all members of the family. In the case of the sloping lines $y = x + c$ mentioned above, this is soon done. Differentiation at once gives $y' = 1$ as the equation of the family bereft of its parameter; and it is conversely true that this relation $y' = 1$ can lead back to no non-differential relation other than $y = x + c$.

The case of the circles in the first quadrant is not so transparent; but you can satisfy yourself by differentiation that

$$c = \frac{x + yy'}{1 + y'},$$

whence $$x - c = \frac{(x - y)y'}{1 + y'}, \quad y - c = \frac{y - x}{1 + y'}.$$

As the circles can be written

$$(x - c)^2 + (y - c)^2 = c^2,$$

we ultimately get

$$(x - y)^2(1 + y'^2) = (x + yy')^2 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

as the differential equation of the family without its parameter. It is of the first order, second degree and non-linear.

The general method for a one-parameter family is now clear. We acquire a second equation by differentiation. As only two equations are needed to eliminate a single parameter, a single differentiation will suffice. We reach the conclusion that a one-parameter family leads to a first-order equation, and the case of the circles suggests that the equation is unlikely to be of the first degree if the parameter is present in powers other than the first. This suggests that conversely the solution of a first-order equation will introduce a single arbitrary parameter.

When two parameters are present, their elimination necessitates three equations. The given relation will serve for one; two others can be derived from it by a first and second differentiation. These three relations serve (in general theory at least) for the elimination of the two parameters, and they lead to a differential equation of the second order. As an example consider our ellipse $(x/a)^2 + (y/b)^2 = 1$ with its two parameters a, b. A first differentiation gives

$$\frac{x}{a^2} + \frac{yy'}{b^2} = 0.$$

A second gives $$\frac{1}{a^2} + \frac{yy'' + y'^2}{b^2} = 0.$$

It happens fortuitously that the original equation plays no part in the elimination. This comes solely from the second and third, and leads to

$$yy' = x(yy'' + y'^2),$$

which is of the second order, first degree and non-linear. It appears that the elimination of two parameters involves two differentiations and introduces y'' so that the resulting equation is of the second order. The suggested converse is that the solution of a second-order equation will involve two parameters or arbitrary constants. You may take it as a rule that the solution of any equation involves a number of constants equal to the order. The proof of such an abstract proposition is the province of the pure mathematician.

1, 7·1. A solution involving the appropriate number of arbitrary constants is sometimes academically referred to as the general solution, or the general integral, or the complete primitive. The solution becomes particular when precise numerical values are given to some or all of the constants. Thus 1, 7 (i) is the general solution of 1, 7 (ii). If we put $c = 0$, we get the point-circle $x^2 + y^2 = 0$ as a particular solution. This last equation involves $x = 0 = y$, and these values obviously satisfy the differential equation 1, 7 (ii).

1, 7·2. The actual values of the constants in any given problem are determined by specified conditions, usually known as boundary conditions, or end conditions, or initial conditions. The stop-watch may be started when a moving light-spot is passing the zero-mark, and we have $x = 0$ when $t = 0$. A loaded built-in beam has zero deflection at the wall and projects horizontally, so that $y = 0$ and $y' = 0$ when $x = 0$, if y be the deflection at distance x from the wall.

1, 7·3. *Summary.*

A differential equation occurs when a rate of change takes place in accordance with a mathematically expressible law. It may also occur by eliminating the parameters from a family of curves; in which case it represents a common property of the family and the order is the number of parameters. Conversely, in solving an equation we expect the appearance of constants, in number equal to the order. Their values are usually determined from specified conditions.

EXERCISES, 1, 8

1. Verify that all circles which touch OY at the origin are included in

$$x^2 + y^2 = 2cx.$$

Deduce that the differential equation of the family is

$$y^2 - x^2 = 2xyy'.$$

2. All parabolas whose vertices touch OY at the origin are included in $y^2 = cx$, whence $y = 2xy'$. The fact that $y' = y/2x$ provides a geometrical method of drawing the tangent at any point.

3. The adiabatic law $pv^\gamma = c$ leads to

$$v\frac{dp}{dv} + \gamma p = 0$$

by eliminating c.

4. The catenary $y = c\cosh\frac{x}{c}$ leads to

$$y\sinh^{-1}y' = x\sqrt{(1 + y'^2)}.$$

5. The family of confocal conics $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$

gives $$y'(a^2 - b^2) = (x + yy')(xy' - y)$$

by the elimination of the parameter λ.

6. A point describing rectilinear simple harmonic motion of frequency $\omega/2\pi$ per second has displacement x at time t given by either

$$x = A\sin\omega t + B\cos\omega t$$

or $$x = R\sin(\omega t + \varphi).$$

The corresponding equation in both cases is

$$\ddot{x} + \omega^2 x = 0.$$

7. In a resisted harmonic motion the displacement may have the form

$$x = Ae^{-\alpha t}\sin(\beta t + \varphi).$$

By eliminating the amplitude A and the phase angle φ, prove that the equation is

$$\ddot{x} + 2\alpha\dot{x} + (\alpha^2 + \beta^2)x = 0.$$

8. If a and b are arbitrary constants, find the equation whose solution is

$$y = a\sec x + b\tan x. \quad [y'' = y\sec^2 x + y'\tan x.]$$

9. The formula $u = Pr + \frac{Q}{r}$ occurs in the theory of the strength of cylinders. Prove that it derives from

$$r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} = u$$

$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(ur)\right\} = 0.$$

10. All rectangular hyperbolas with their asymptotes parallel to the co-ordinate axes are contained in $2y'y''' = 3y''^2$.

11. The relation

$$y = A \sin ax + B \cos ax + C \sinh ax + D \cosh ax$$

is of frequent occurrence. It corresponds to

$$\frac{d^4y}{dx^4} = a^4y.$$

12. As a departure from the routine procedure, consider a parabola whose focus is at the origin O and whose vertex is on OX left of the origin. If P be any point on this curve (in the first quadrant for convenience) and the tangent PT makes an angle θ with OX, it is a known property of the curve that OP is inclined at 2θ to OX. As $\tan\theta = y'$ by the elementary calculus and $\tan 2\theta = y/x$ from the figure, the trigonometrical relation between $\tan\theta$ and $\tan 2\theta$ leads to the differential equation of the family. You can verify your result after satisfying yourself that the family has the cartesian equation $y^2 = 4c(x + c)$, leading to

$$y(1 - y'^2) = 2xy'.$$

13. A procedure similar to No. 12 can be applied to the circles touching both axes; see 1, 7 (i) and (ii). Let P, with co-ordinates x, y, be any point on one of the circles. Let T be the foot of the perpendicular from the origin O on to the tangent at P. It can be shown by elementary geometry that $PT = x - y$. If PT, TO be inclined at ψ, α respectively to OX, we have $\tan\psi = y'$ and $\psi = \frac{1}{2}\pi + \alpha$. This leads to

$$PT = x - y = y\cos\alpha - x\sin\alpha = y\sin\psi + x\cos\psi,$$

and the differential equation of the family follows.

14. The primitive $y = ae^{bx+c}$ leads to

$$\frac{d^2}{dx^2}(\log y) = 0.$$

How do you explain this being of the second order, whereas there are apparently three constants?

1, 9. *Hyperbolic Functions.*

For those who have the right type of mind the solution of differential equations is an end in itself, and is capable of giving considerable mental satisfaction. But for the practical man it is essential that the solution should be translatable into figures. It is presumed in the present work that the reader is prepared to obtain numerical results. These need not be carried to a high degree of accuracy, so that a set of four-figure tables will suffice. Even better is a slide-rule fitted with an exponential scale. The solution of such equations as

$$x^{2 \cdot 3} = 8 \cdot 5, \quad \text{or} \quad 3 \cdot 9^x = 11 \cdot 2$$

can then be picked off in less time than it takes to find the right page in the tables. Moreover, the logarithms that appear naturally are Napierian, and few small books contain satisfactory tables of these. Hyperbolic functions crop up repeatedly, and here the small tables are even less satisfactory.

Experience shows that these simple functions are not usually as well known as is desirable, especially in their integral properties. In many ways they are simpler than plane trigonometry and they are easily tamed with the aid of a slide-rule. Whether we like it or not they make their appearance in scientific investigations quite naturally, and it is as well to dominate them at the outset. The next few pages are intended to facilitate this.

1, 10. *The Exponential Theorem.*

We begin with the exponential theorem. The number which is universally denoted by e makes a natural appearance in mathematics in the form of an infinite and convergent series,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots .$$

The symbol $n!$, read "factorial n", denotes the continued product of the integers from 1 to n inclusive, so that $5! = 1 . 2 . 3 . 4 . 5 = 120$. The value of e has been somewhat uselessly computed to several hundred places of decimals, the first few figures being 2·7182818... . The decimal neither terminates nor recurs, so that e is incommensurable and cannot be succinctly and accurately represented by a fraction with integral numerator and denominator.

The exponential theorem states that for a positive x we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

The number of terms is infinite, and the series converges for all values of x. The sum is essentially positive since each term is positive. As

x approaches zero the value approaches unity since $e^0 = 1$. As x becomes large, the value increases without limit; no number N is so large but that an x can be found to make e^x larger than N.

For negative values we have

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots .$$

This converges for all values of x and is essentially positive since e^{-x} is the reciprocal of e^x. As x becomes large e^x increases indefinitely, so that e^{-x} tends to zero whilst remaining positive.

The curve $y = e^x$ thus lies wholly above OX, and is asymptotic to it on the left, crossing OY at 45° with intercept unity. The curve $y = e^{-x}$ is its mirror reflection in OY, so that the curves are perpendicular where they cross at (0, 1). Any horizontal line above OX cuts each of these curves once only; any horizontal line below OX never cuts either of them. The equation $e^x = a$ therefore always has one real root for any positive a, large or small. There are no real roots if a is negative; the solution of such an equation as $e^x + 2{\cdot}31 = 0$ is imaginary. It is worth noting that e^x is frequently written $\exp x$ if the exponent x is at all complicated.

1, 11. *Definitions and Properties.*

The hyperbolic functions are defined in analogy with the exponential values of the trigonometrical functions. Just as we write

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}),$$

so we write $\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \quad \sinh x = \tfrac{1}{2}(e^x - e^{-x}).$

Two things to notice are, (i) the imaginary i makes no appearance in the hyperbolic definitions; (ii) where the number θ is an angle in radians, x is pure number and has no connexion with angle.

The definitions are equivalent to

$$\cosh x + \sinh x = e^x,$$
$$\cosh x - \sinh x = e^{-x}.$$

We can immediately make observations on their properties. Cosh x, being the half-sum of two positive numbers, is essentially positive, and it is easily shown to have a minimum value of unity when x is zero; $\cosh 0 = 1$. It is a symmetrical or even function in the sense that $\cosh(-x) = \cosh x$; consequently an equation like $\cosh x = 3{\cdot}8$

has two equal and opposite roots, whilst equations like $\cosh x = \frac{1}{2}$, or $\cosh x = -1{\cdot}3$, have no real roots at all.

Sinh x is a skew or odd function since $\sinh(-x) = -\sinh x$. The curve $y = \sinh x$ ranges from $+\infty$ when x is large and positive, to $-\infty$ when x is large and negative. It inflects through the origin at 45° and $\sinh 0 = 0$. Any horizontal line meets it once, so that the equation $\sinh x = a$ always has a real root if a is real, the two being positive or negative together.

In the relation

$$\cosh x - \sinh x = e^{-x},$$

the right side is essentially positive, so that $\cosh x > \sinh x$ for all values of x. For large positive values of x, the values of $\cosh x$ and $\sinh x$ are practically equal and indistinguishable from $\frac{1}{2}e^x$.

The product of the two relations gives

$$(\cosh x + \sinh x)(\cosh x - \sinh x) = e^x \cdot e^{-x} = e^0,$$

or

$$\cosh^2 x - \sinh^2 x = 1,$$

which again shows that $\cosh x > \sinh x$.

It can be left to the reader to prove that each of these functions is the indefinite integral of the other and also the differential coefficient of the other.

$$\frac{d}{dx}\cosh x = \sinh x = \int \cosh x\, dx,$$

$$\frac{d}{dx}\sinh x = \cosh x = \int \sinh x\, dx.$$

Their series are

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots,$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots,$$

all signs being plus, so that for small values of x we have the approximations

$$\cosh x \doteq 1 + \tfrac{1}{2}x^2, \quad \sinh x \doteq x.$$

The function $\tanh x$ is given by

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Since $\sinh x$ is less than $\cosh x$, $\tanh x$ is less than unity in absolute value and takes the sign of x. It is a skew function since $\tanh(-x) = -\tanh x$. The curve $y = \tanh x$ passes through the origin at 45° with inflection; $\tanh 0 = 0$, and $\tanh x$ is always less than x in absolute value. On the right it is asymptotic to $y = 1$, and on the left it is asymptotic to $y = -1$. It follows that the equation $\tanh x = a$ has a real root whenever a is less than unity in absolute value; otherwise the root is imaginary.

The chief properties of $\operatorname{sech} x$ can be deduced from $\operatorname{sech} x = 1/\cosh x$. Its value is always positive and it is symmetrical. It has a maximum on OY; $\operatorname{sech} 0 = 1$. It is asymptotic to OX at both ends, and hence it must have two inflections. These two functions are related by

$$\frac{d}{dx} \operatorname{sech} x = -\tanh x \operatorname{sech} x,$$

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x.$$

Note that the former differs in sign from its trigonometrical analogue.

The other two functions are more rarely used. $\operatorname{Cosech} x = 1/\sinh x$ is doubly asymptotic to both OX and OY, and looks rather like the rectangular hyperbola $xy = 1$; it has two separate branches in the first and third quadrants.

$\operatorname{Coth} x = 1/\tanh x$ likewise has two separate branches in the first and third quadrants. Both branches are asymptotic to OY; but the one is asymptotic to $y = 1$ whereas the other is asymptotic to $y = -1$. The branch of $\coth x$ in the first quadrant lies wholly above the corresponding branch of $\operatorname{cosech} x$.

The hyperbolic functions find their first application as parametric co-ordinates on the hyperbola. If $(x/a)^2 - (y/b)^2 = 1$, we can take

$$x = a \cosh u, \quad y = b \sinh u.$$

Note at the same time that we can equally well take

$$x = a \sec \theta, \quad y = b \tan \theta.$$

At any point L on OX erect the ordinate to meet the curve at A and the asymptote at B. The geometrically obvious fact that LB is greater than LA is equivalent to $\cosh u > \sinh u$. Also if C be the vertex of the curve, the area of the sector OCA is $\frac{1}{2}abu$, just as in the case of an ellipse the corresponding area would be $\frac{1}{2}ab\,\theta$, where θ is the eccentric angle. The result is easily established by the formula

$$\text{sector } OCA = \tfrac{1}{2}\int_{C}^{A} (x\,dy - y\,dx).$$

1, 12. *Inverse Functions.*

The direct and inverse functions are merely two ways of looking at the same graph. If a set of tables gives $\cosh 1{\cdot}4 = 2{\cdot}15$, this is a succinct way of saying that in the graph of $y = \cosh x$ an abscissa of 1·4 gives an ordinate of 2·15. If we choose to view the statement the other way round and say "an ordinate of 2·15 gives an abscissa of 1·4", then 1·4 is some function of 2·15 and it is convenient to write it as $1{\cdot}4 = \cosh^{-1} 2{\cdot}15$. There is nothing in the inverse functions more than that; in fact, the graph $y = \cosh x$ becomes $y = \cosh^{-1} x$ if we rotate the figure through $\frac{1}{2}\pi$ clockwise. The same is true of the sech curve, but not exactly of the sinh or tanh curves since rotation would need to be followed by reflection in the new OX.

1, 13. *Numerical.*

In handling the direct functions $\cosh x$ and $\sinh x$ on a slide-rule, we begin by picking off e^x and then reciprocating it. The half sum or difference then gives $\cosh x$ or $\sinh x$. If both are required, it is quicker after first calculating $\cosh x$ to get $\sinh x$ by subtraction, from the formula

$$\sinh x = \cosh x - e^{-x}.$$

Example: $x = 0{\cdot}68$; $e^x = 1{\cdot}973$.

Reciprocal $e^{-x} = 0{\cdot}507$; sum $= 2{\cdot}480$.

$\frac{1}{2}$ sum $= 1{\cdot}240 = \cosh 0{\cdot}68$.

Difference $= 0{\cdot}733 = \sinh 0{\cdot}68$.

In the evaluation of $\tanh x$ it is better to employ

$$\tanh x = (e^{2x} - 1)/(e^{2x} + 1).$$

Example.—Calculate tanh 0·37.

$$x = 0{\cdot}37,\ 2x = 0{\cdot}74,\ e^{2x} = 2{\cdot}09.$$

$$\tanh x = \frac{2{\cdot}09 - 1}{2{\cdot}09 + 1} = 0{\cdot}353,$$

a result that accords with the statement that $\tanh x < x$.

The determination of the inverse functions $\cosh^{-1} y$ and $\sinh^{-1} y$ depends on $\cosh x + \sinh x = e^x$. Whichever is given, the other has to be found and the answer deduced from the exponential. An example will make this clear.

Example.—Find $\sinh^{-1} 1{\cdot}37$.

Put $\sinh^{-1} 1{\cdot}37 = x$, so that

$$\sinh x = 1{\cdot}37,\ \sinh^2 x = 1{\cdot}89,$$
$$\cosh^2 x = 2{\cdot}89,\ \cosh x = 1{\cdot}70,$$
$$\text{sum} = 3{\cdot}07 = e^x,$$
$$\text{whence } x = 1{\cdot}12 = \sinh^{-1} 1{\cdot}37.$$

For the inverse function $\tanh^{-1} y$ we employ an earlier relation in the form

$$e^{2x} = \frac{1 + \tanh x}{1 - \tanh x}.$$

Example.—Calculate $\tanh^{-1} 0{\cdot}88$.

Put $\tanh^{-1} 0{\cdot}88 = x$, so that

$$\tanh x = 0{\cdot}88,\ e^{2x} = \frac{1{\cdot}88}{0{\cdot}12} = 15{\cdot}7,$$
$$2x = 2{\cdot}76, \quad x = 1{\cdot}38 = \tanh^{-1} 0{\cdot}88.$$

1, 14. *Standard Integrals.*

The hyperbolic functions play a useful part in uniformizing certain standard forms in the integral calculus.

When $a > x$, we have $\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a}$ or $-\cos^{-1}\frac{x}{a}$.

This double form of the result is at first rather mystifying; but the explanation is simple. For any angle β we have $\cos\beta = \sin(\frac{1}{2}\pi - \beta)$. If we put each of these equal to v, we have $\beta = \cos^{-1} v$,

$$\tfrac{1}{2}\pi - \beta = \sin^{-1} v = \tfrac{1}{2}\pi - \cos^{-1} v.$$

In the indefinite integral, the $\frac{1}{2}\pi$ is omitted as being merely part of the arbitrary integration-constant.

Taking the allied form $\int \frac{dx}{\sqrt{(x^2 - a^2)}}$, $x > a$,

put $x = a\cosh u$ so that $dx = a \sinh u\, du$, $\sqrt{(x^2 - a^2)} = a \sinh u$. The integral becomes

$$\int du = u = \cosh^{-1}\frac{x}{a}.$$

The substitution $x = a \sinh u$ gives $dx = a \cosh u\, du$, $\sqrt{(x^2 + a^2)} = a \cosh u$. Hence

$$\int \frac{dx}{\sqrt{(x^2 + a^2)}} = \int du = u = \sinh^{-1} \frac{x}{a}.$$

Three other forms can be similarly treated. We have the standard form

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad \text{or} \quad -\frac{1}{a} \cot^{-1} \frac{x}{a}.$$

The substitution $x = a \tanh u$ gives

$$dx = a \operatorname{sech}^2 u\, du, \quad (a^2 - x^2) = a^2 \operatorname{sech}^2 u.$$

Hence $$\int \frac{dx}{a^2 - x^2} = \int \frac{du}{a} = \frac{u}{a} = \frac{1}{a} \tanh^{-1} \frac{x}{a}.$$

The substitution $x = a \coth u$ gives

$$dx = -a \operatorname{cosech}^2 u\, du, \quad (x^2 - a^2) = a^2 \operatorname{cosech}^2 u.$$

Hence $$\int \frac{dx}{x^2 - a^2} = -\frac{u}{a} = -\frac{1}{a} \coth^{-1} \frac{x}{a}.$$

These last two forms are essentially the same; they merely differ in sign. As $\tanh u$ is less than unity in absolute value, whilst $\coth u$ is greater than unity in absolute value, the two results hold in different regions according as x is less or greater than a.

Consider the integral

$$\int \sec x\, dx = \log (\sec x + \tan x),$$

a result which always has to be looked up in the book and whose proof involves a feat of memory. We have previously mentioned the two-fold parametric form of the co-ordinates on a hyperbola. They suggest putting $\sec x = \cosh u$, whence $\tan x = \sinh u$ and $\sec x \tan x\, dx = \sinh u\, du$, so that $\sec x\, dx = du$. We then have

$$\int \sec x\, dx = \int du = u = \sinh^{-1} \tan x = \cosh^{-1} \sec x.$$

In a similar manner the equivalent substitutions $\cot x = \sinh u$, $\operatorname{cosec} x = \cosh u$ gives

$$\int \operatorname{cosec} x\, dx = -\cosh^{-1} \operatorname{cosec} x = -\sinh^{-1} \cot x.$$

The above results are assembled below for convenience of reference.

It remains to point out the connexion between hyperbolic and trigonometric functions, which is

$$\cos ix = \cosh x, \qquad \cosh ix = \cos x,$$
$$\sin ix = i \sinh x, \qquad \sinh ix = i \sin x.$$

Note that $\cosh \log x = \frac{1}{2}(x + x^{-1})$ with other similar results.

1, 15. *Standard Forms.*

$$a > x, \int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1} \frac{x}{a} \text{ or } -\cos^{-1} \frac{x}{a}.$$

$$x > a, \int \frac{dx}{\sqrt{(x^2 - a^2)}} = \cosh^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{\sqrt{(x^2 + a^2)}} = \sinh^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \text{ or } -\frac{1}{a} \cot^{-1} \frac{x}{a}.$$

$$a > x, \int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}.$$

$$x > a, \int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \coth^{-1} \frac{x}{a}.$$

$$\int \sec x \, dx = \sinh^{-1} \tan x \text{ or } \cosh^{-1} \sec x.$$

$$\int \operatorname{cosec} x \, dx = -\sinh^{-1} \cot x \text{ or } -\cosh^{-1} \operatorname{cosec} x.$$

It is well worth the student's while to draw the graphs of these functions. A suitable range of values is $-3{\cdot}5 < x < 3{\cdot}5$ at intervals of 0·5. The horizontal axis should be taken rather below the middle of the paper since the functions lie for the most part above OX. The appearance is improved if the horizontal scale is made rather larger than the vertical scale. A suitable unit for y is 2 cm. and for x 2·5 cm.

CHAPTER II

First Order; Standard Methods

2, 1. The equation of the first order and first degree is one of the simplest classes of differential equations and is fortunately one of the most useful. It can be written as $dy/dx = \phi(x, y)$. It is not soluble in general; if $\phi(x, y)$ be written down at random it is highly unlikely that the resulting equation would prove tractable. The soluble cases fall into certain well-defined types which are classified as:

(i) variables separable,
(ii) the homogeneous form,
(iii) the exact equation,
(iv) the linear equation.

2, 2. *Variables Separable.*

The variables are said to be separable if the equation can be written with one side as a function of x, multiplied by dx, whilst the other side can correspondingly be written as a function of y, multiplied by dy. It follows that $\phi(x, y)$ will have one of the five forms $P(x)$, $Q(y)$, $P(x) \, . \, Q(y)$, $P(x)/Q(y)$, $Q(y)/P(x)$. When the variables have been separated, we can write

$$f(x)\,dx = F(y)\,dy.$$

In particular cases, either $f(x)$ or $F(y)$ may be a mere constant. Integration at once gives

$$\int f(x)\,dx = \int F(y)\,dy + \text{constant}.$$

This is considered to be the solution even if the integration cannot be performed in finite terms. The constant is determined from assigned conditions, or otherwise left arbitrary.

2, 3. The following examples illustrate the method.

Example 1.—Solve $x\dfrac{dy}{dx} = \cot y$ with the conditions that $y = \pi/4$ when $x = 1$.

We have $$\int dx/x = \int \tan y\,dy + \text{constant},$$

whence $$\log x = \log \sec y + \log c.$$

Here the constant is taken as $\log c$ since both the other terms are logarithms. $\log c$ is just as arbitrary as c, and this procedure usually adds to the elegance of the solution. This gives $x = c \sec y$. If $x = 1$ when $y = \pi/4$, substitution gives $1 = c \sec \pi/4$, so that $c = \cos \pi/4 = 1/\sqrt{2}$. The solution thus becomes $\sec y = x\sqrt{2}$.

Certain simple electrical problems can be solved by this method.

Example 2.—A condenser of capacity C, charged to a voltage V_0, is discharged through a resistance R. Find the voltage V at any subsequent time t.

The charge Q on the condenser, measured in coulombs, is given by $Q = CV$. The current I is defined as the rate of change of Q. This is negative as the condenser is discharging, and we have

$$I = -\frac{dQ}{dt} = -C\frac{dV}{dt}.$$

Ohm's law, that resistance × current = voltage, then gives

$$RI = V = -RC\frac{dV}{dt},$$

and the variables are separable:

$$\frac{dV}{V} = -\frac{dt}{RC}$$

and

$$\log V = \log a - \frac{t}{RC}.$$

The arbitrary a is identifiable with V_0 since that is the voltage when t is zero if we count the time from the closing of the circuit. This leads to

$$V = V_0 \exp(-t/RC).$$

The fact that V is defined by a negative exponential shows that it asymptotically approaches zero with the lapse of time. We examine the matter numerically. The farad unit of capacity is too large for practical purposes; the microfarad, μF, or millionth of a farad, is commoner. Suppose, then, that the condenser has $C = 0{\cdot}3$ μF and $R = 20$ ohms, in what time will the condenser lose 99·9 per cent of its charge? We have

$$RC = 20 \times 3 \,.\, 10^{-7} = 6 \,.\, 10^{-6}; \; V/V_0 = 10^{-3} = \exp(-t/RC),$$

since V is only 0·1 per cent of V_0.

$$t/RC = \log_e 10^3 = 6{\cdot}91; \; t = 4 \,.\, 10^{-5} \text{ sec. approximately.}$$

Example 3.—A series circuit consists of a battery of constant voltage E, a resistance R, and an inductance L. If the circuit, initially open, is closed at time $t = 0$, calculate the rise of the current.

The "voltage" to be used in Ohm's law is the total voltage, consisting here of a battery E and a back E.M.F. $= -L\,dI/dt$ from the inductance. Hence

$$RI = E - L\frac{dI}{dt}$$

and the variables are separable.

$$\frac{dt}{L} = \frac{dI}{E - RI},$$

whence
$$\frac{t}{L} = -\frac{1}{R}\log(E - RI) + \frac{1}{R}\log c.$$

Here the arbitrary constant has been given the convenient form $(\log c)/R$. Initially there is no current, so $t = 0 = I$ gives $c = E$.

$$\frac{t}{L} = \frac{1}{R}\log\frac{E}{E - RI}.$$

This gives the time t in terms of the current I. It is customary to reverse the relation by writing

$$\exp\left(\frac{Rt}{L}\right) = \frac{E}{E - RI},$$

whence by inversion
$$\exp\left(-\frac{Rt}{L}\right) = 1 - \frac{RI}{E},$$

so that
$$I = \frac{E}{R}\left\{1 - \exp\left(-\frac{Rt}{L}\right)\right\}.$$

As t becomes large the exponential goes towards zero asymptotically, so that the current I asymptotically approaches its steady final value E/R. The unit of inductance is the henry, but in practice the millihenry is commoner. It is written mH, and its value is one-thousandth of a henry, or 10^{-3} H $= 1$ mH. To get an idea of the order of magnitude of events, suppose a battery (whose voltage is immaterial) acts on a 20-ohm coil whose inductance is 0·03 mH. The current reaches 99·9 per cent of its final value when $1 - \exp(-Rt/L) = 0{\cdot}999$. This gives $\exp(-Rt/L) = 0{\cdot}001$; or $\exp(Rt/L) = 10^3$. Hence

$$\frac{20t}{3 \,.\, 10^{-5}} = \log_e 10^3 = 6{\cdot}908$$

and $t = 10^{-5}$ sec. approximately, or a hundred thousandth of a second.

The following is an application of the method to a geometrical problem.

Example 4.—Find a curve in which the projection of the ordinate on the normal is constant.

If the tangent at any point of a curve makes an angle ψ with OX, this is the angle between the ordinate and the normal. By hypothesis the curve has $y\cos\psi = a$, or $\cos\psi = a/y$, where a is the constant length of the projection. But

$$dy/dx = \tan\psi = \sqrt{y^2 - a^2}/a.$$

Hence, on separation,
$$\frac{dy}{\sqrt{(y^2 - a^2)}} = \frac{dx}{a}$$

and
$$\cosh^{-1}\frac{y}{a} = \frac{x}{a} + c.$$

If $x = 0$ when $y = a$, then $c = 0$ since $\cosh^{-1} 1 = 0$. Hence the curve is

$$y = a \cosh \frac{x}{a},$$

the common catenary.

2, 3·1. *Loss of Solutions.*

The above illustrates a point which is frequently overlooked. When the separation of the variables involves division it occasionally happens that some of the solutions are lost, in the sense that they cannot be derived from the general solution by giving particular values to the arbitrary parameter. If in the above Ex. 4 we put $y = \pm a$, then $dy/dx = 0$, and as these identically satisfy the differential equation they must be part of the solution; but they cannot be derived from the general solution by giving a value to the parameter c.

2, 4. The separation of the variables is of frequent use in solving mechanical problems.

Example 5.—A body of mass m falls vertically by the action of gravity. It experiences an atmospheric resistance kv^2 when its velocity is v. Discuss the motion.

Taking the downward direction to be positive as measured from some initial height, the mass has acceleration dv/dt. The downward force is mg; the atmospheric resistance upwards is kv^2. The equation of downward motion is

$$mg - kv^2 = m\frac{dv}{dt}.$$

The variables are separable; we can write

$$k\,dt = \frac{m\,dv}{a^2 - v^2},$$

where $a^2 = mg/k$. Integration gives (if $v < a$)

$$c + kt = \frac{m}{a}\tanh^{-1}\frac{v}{a}.$$

If we suppose the motion to have started from rest, we have $t = 0 = v$. Substitution gives $c = (m/a)\tanh^{-1} 0$, so that c is zero.

$$\frac{v}{a} = \tanh\frac{akt}{m} = \tanh bt,$$

where $b = ak/m = \sqrt{(gk/m)}$. The function $\tanh bt$ is asymptotic to unity with increasing t. It follows that the velocity asymptotically approaches the value $a = \sqrt{(mg/k)}$. This is known as the "terminal velocity". The mass would cease to accelerate when the atmospheric resistance just counterbalanced the gravitational pull, a state of affairs which is gradually approached as the velocity of descent increases. For this reason the arrival velocity of raindrops or hailstones is little indication of the height from which they have fallen, and a parachutist reaches earth with a reasonable landing velocity. Numerically, if the body reached 99·9 per cent of its terminal velocity in 7·6 seconds, we can get some light on k. We have

$$\tanh bt = 0{\cdot}999 = \frac{\exp(2bt) - 1}{\exp(2bt) + 1},$$

$$\exp(2\,bt) = 1999,\ 2\,bt = \log_e 1999 = 7{\cdot}60,$$

$$b = \frac{7{\cdot}60}{2 \times 7{\cdot}60} = 0{\cdot}5 = \sqrt{(gk/m)},$$

$$k/m = 2{\cdot}55 \times 10^{-4},$$

the unit being cm.$^{-1}$.

[It is suggested that the reader now try some of the exercises, postponing the next three sections till a second reading.]

2, 4·1. *Interpretation of the Form.*

A glimpse of the inner meaning of this soluble form can be obtained by considering a family of curves which is generated by translating a given curve; and here the word "translating" is used in its kinematical sense as being devoid of rotation. Consider the curve $y = f(x)$; if we transfer the origin a distance c to the right, the equation becomes $y = f(x + c)$. The same result would be obtained by keeping the axes fixed and translating the curve a distance c to the left. The different curves which can be obtained by varying the value of c form a one-parameter family. All members of this family are identical in shape, size and orientation. Any one can be made to coincide with any other by lateral translation. If the original curve be equivalent to $x = F(y)$, the family $y = f(x + c)$ can be written $x + c = F(y)$. The differential equation of this family is $1 = F'(y)\,dy/dx$, whose variables are separable. If we write $P(y)$ for the reciprocal of $F'(y)$, we conclude that the equation $dy/dx = P(y)$ represents a one-parameter family of curves formed by translating some curve along the axis OX. Similarly, $dy/dx = Q(x)$ represents a one-parameter family formed by pure translation along OY.

2, 4·2. *Oblique Translation.*

This raises the question of what happens if the translation is oblique. Suppose A, B are two points on the curve $y = f(x)$. Let us form a family by translating this curve along $ax + by = 0$. The point A will take other positions A_1, A_2, &c., and these will all lie along a parallel line $ax + by = c_1$. Similarly, B takes positions B_1, B_2, &c., which lie on a line $ax + by = c_2$. The tangents at A, A_1, A_2, &c., are all parallel to each other; and likewise the tangents at B, B_1, &c. A rough sketch will make this clear. It appears that along any line $ax + by = c$ parallel to the direction of translation, the value of dy/dx is constant; but it varies from line to line, i.e. it changes with c. The mathematical expression of this fact is $dy/dx = F(c)$, or $dy/dx = F(ax + by)$, since $ax + by = c$.

This is about the only way of disguising the form which is soluble by separation of the variables. Even so, the disguise is easily penetrated; we can solve by taking a new dependent variable defined by $t = ax + by$.

Example 6.—Solve $\dfrac{dy}{dx} = \sin^2(x - 2y)$.

Put $t = x - 2y$, so that

$$\frac{dt}{dx} = 1 - 2\frac{dy}{dx} = 1 - 2\sin^2(x - 2y) = \cos 2t.$$

The equation becomes $\sec 2t\, dt = dx$,

so that $\cosh^{-1}(\sec 2t) = 2x + c$,

and $\sec 2(x - 2y) = \cosh(2x + c)$,

or $\tan 2(x - 2y) = \sinh(2x + c)$,

or $\sin 2(x - 2y) = \tanh(2x + c)$.

2, 4·3. *Generalized Translation.*

To complete the picture we consider a sort of generalized translation. The co-ordinates of a point on a curve are often given parametrically; thus for an ellipse we often take $a\cos\theta$, $b\sin\theta$; or for a parabola $y^2 = 4ax$ we take at^2, $2at$. Suppose a curve given by $x = f_1(\theta)$, $y = f_2(\theta)$. We can create a one-parameter family from this by writing

$$x = f_1(\theta), \quad y = f_2(\theta + c).$$

The elimination of θ would give $\phi(x, y, c) = 0$ as our one-parameter family. If we suppose that the original parametric equations can be inverted into

$$F_1(x) = \theta = F_2(y),$$

the amended equations for the family give

$$\theta = F_1(x), \quad \theta + c = F_2(y);$$

so that our family is $F_2(y) = F_1(x) + c.$

The differential equation of this family is

$$F_2'(y)\frac{dy}{dx} = F_1'(x),$$

whose variables are separable. Notice that the parametric form of the original curve can be recovered by writing

$$F_2'(y)\,dy = d\theta = F_1'(x)\,dx.$$

We conclude that any equation of the form $P(x)\,dx = Q(y)\,dy$ represents a one-parameter family which has been formed by varying one of the co-ordinates of the curve when expressed parametrically.

Example 7.—Consider the curve defined by $x = \cos \omega t$, $y = \sin 2\omega t$. If t be interpreted as the time, whilst ω is a constant, it represents a point performing simultaneously two perpendicular simple harmonic motions. Their amplitudes are the same, but the vertical motion is twice as rapid as the horizontal. This usually gives a Lissajous figure of eight. The curve in this case is $2\cos^{-1}x = \sin^{-1}y$. We can throw one of the motions out of phase by writing

$$x = \cos(\omega t + c), \quad y = \sin 2\omega t.$$

The family is now $2\cos^{-1}x = \sin^{-1}y + 2c$. Its differential equation is

$$-\frac{2dx}{\sqrt{(1-x^2)}} = \frac{dy}{\sqrt{(1-y^2)}},$$

and the variables are separated. To recover the parametric form of the original curve, we write

$$-\frac{2dx}{\sqrt{(1-x^2)}} = d\theta = \frac{dy}{\sqrt{(1-y^2)}},$$

whence $$2\cos^{-1}x = \theta = \sin^{-1}y,$$

or $y = \sin\theta$, $x = \cos\frac{1}{2}\theta$, the only difference being that θ has replaced $2\omega t$.

EXERCISES, 2, 5

1. $\dfrac{dy}{dx} = x^2y^2$. [$y(a - x^4) = 4$.]

2. $\dfrac{dy}{dx} = \cot x \cot y$. [$\sin x \cos y = c$.]

3. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$. [$\tan x \tan y = c$.]

4. $\dfrac{1}{y}\dfrac{dy}{dx} = \tanh x - \tan x$. [$\cos x \cosh x = cy$.]

5. $\dfrac{dy}{dx}\sqrt{(a + x)} + x = 0$. [$2(x - 2a)\sqrt{(x + a)} = c - 3y$.]

6. $\dfrac{dy}{dx} = \sqrt{(a^2 - y^2)}$. [$y = a \sin(x + c)$.]

Note the loss of the solution $y = a$.

7. $\dfrac{dy}{dx} + \left(\dfrac{1 - y^2}{1 - x^2}\right)^{\frac{1}{2}} = 0$.

Obtain the solution in algebraic form. [$y\sqrt{(1 - x^2)} + x\sqrt{(1 - y^2)} = c$.]

8. Prove that $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}$ represents rectangular hyperbolas which all pass through the points (1, 1) and (−1, −1).

9. Solve the equation $x\dfrac{dy}{dx} - y = xy$, and determine the arbitrary constant so that the curve passes through the origin at 45° to OX. [$y = xe^x$.]

10. Prove that no solution of $\dfrac{dy}{dx} + (y - 1)\cos x = 0$ can cross or touch the line $y = 1$. Any solution must oscillate between two horizontal lines which are both above or both below $y = 1$. Sketch a few members of the family.

[$y = 1 + a \exp(-\sin x)$.]

11. Find a curve in which the normal is proportional to the square of the ordinate. Determine the constant so that OY is crossed horizontally.

[A catenary.]

12. Find a curve whose normal is of constant length.

[A circle with centre on OX.]

13. A simple series circuit consists of a resistance R, a battery of constant voltage E and a condenser of capacity C. Initially the circuit is open and there is no charge on the condenser. If V be the voltage of the condenser at time t after closing the circuit, prove that V asymptotically approaches its final value E in accordance with $V = E\left\{1 - \exp\left(-\dfrac{t}{RC}\right)\right\}$.

If $R = 20$ ohms and $C = 0{\cdot}03$ μF, estimate the time for the condenser to receive 99 per cent of its final charge.

14. A battery of constant voltage E is sending a steady current E/R through a coil of resistance R and inductance L. If the circuit be broken, prove that the current decays to zero by the law $I = \dfrac{E}{R}\exp\left(-\dfrac{Rt}{L}\right)$.

15. An experiment was made to test the insulation of an electric cable. A part of the cable, with capacity 0·237 μF, was charged to 200 volts. The charge escaped through the insulation and in 15 sec. the voltage fell to 163 volts. Calculate the resistance of the insulation in megohms. [313, approximately.]

16. Find the isothermal relation for a gas whose elasticity is proportional to the pressure. The elasticity is defined as the negative rate of change of pressure with volume per unit volume.

[PV^n = constant.]

17. A soluble sphere is placed in liquid and begins to dissolve, but remains spherical. The rate at which it dissolves is proportional to its surface area, but inversely proportional to the strength of the solution. If 10 per cent of the mass dissolves in 10 sec., in what time will 20 per cent have dissolved? [About 42 sec.]

18. Equations of the type $\frac{dx}{dt} + ax^n = 0$ are met in the theory of mass action in physical chemistry; n is necessarily an integer. If corresponding values are

$$x = 1{\cdot}0 \qquad 0{\cdot}5 \qquad 0{\cdot}3,$$
$$t = 0 \qquad 3 \qquad 10,$$

prove that the most likely value for n is 3.

19. A circular column of variable section supports a dead-load P. The mean compressive stress on any horizontal section has the constant value f. The material of construction weighs w per unit volume. If the column has radius x at depth y from the top, prove that

$$\pi f x^2 = P \exp\left(\frac{wy}{f}\right).$$

20. A mass m is projected vertically upward from the ground with initial velocity u. The atmospheric resistance is kv^2 when the velocity is v. Prove that the time of ascent is $t = \frac{a}{g}\tan^{-1}\frac{u}{a}$ where $a^2 = \frac{mg}{k}$.

21. In example No. 5 in the text, taking the acceleration in the form $v\,dv/dx$, where x is the displacement, prove that

$$\frac{2kx}{m} = \log\frac{a^2}{a^2 - v^2},$$

on the assumption that the mass starts from rest with zero displacement. Deduce that if the mass reaches 90 per cent of its terminal velocity in a fall of 100 metres, $k/m = 8{\cdot}3 \times 10^{-5}$ in c.g.s. units. What are the dimensions?

22. In the case of shooting flow of water over a weir, the profile of the free surface satisfies the equation $\frac{dy}{dx} = k\frac{y^3 - a^3}{y^3 - b^3}$. Integrate it.

23. The axes of co-ordinates are the principal axes of a central conic. Prove that its differential equation is $yy' = x(yy'' + y'^2)$. Conversely, prove that this equation gives a central conic. Begin by choosing a new variable defined by $u = yy'$.

24. A moving point with co-ordinates x, y has velocities $(a + x)$ parallel to OY and $(b - y)$ parallel to OX. Prove that it describes a circle. If the velocities be interchanged, prove that the path is a hyperbola.

25. A solution of the equation $\frac{dy}{dx} = (x + y)^2$ passes through the point $(\frac{1}{2}, \frac{1}{2})$. Calculate y when $x = 0{\cdot}7$. [$y = 0{\cdot}8084.$]

26. Prove that the solution of the equation $\frac{dy}{dx} = \sqrt{(x + y)}$ is

$$x + 2\log\{1 + \sqrt{(x + y)}\} = 2\sqrt{(x + y)} + c.$$

27. Prove that the equation

$$\frac{du}{dv} + \frac{b}{a} = \frac{F(Av + C)}{f(au + bv + c)}$$

is soluble by separation of the variables.

28. A one-parameter family is formed from an ellipse by slightly changing the parametric co-ordinates from $x = a\cos\theta$, $y = b\sin\theta$ to $x = a\cos(\theta + \epsilon)$, $y = b\sin\theta$. What does the family look like? What is its cartesian equation and its differential equation? Prove that this last is soluble by separation of the variables.

2, 6. *The Homogeneous Equation.*

Suppose the first-order first-degree equation to be written in the form $M\,dy = N\,dx$ where M, N are functions of x and y. If these are homogeneous and of the same degree the equation is said to be homogeneous. For example, the equation

$$(xy - x^2)\,dy = 2y^2\,dx$$

is homogeneous. The same thing is sometimes expressed by saying that the equation is homogeneous if it can be expressed in the form

$$\frac{dy}{dx} = \phi\left(\frac{y}{x}\right). \qquad \text{(i)}$$

Thus the previous equation could be written as

$$\frac{dy}{dx} = \frac{2\left(\dfrac{y}{x}\right)^2}{\dfrac{y}{x} - 1}.$$

Such an equation can always be solved by choosing a new dependent variable v defined by $y = vx$. This substitution enables us to separate the variables. We have

$$y = vx, \quad \frac{y}{x} = v,$$

so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substituting these values in equation (i), we have

$$\phi(v) = v + x\frac{dv}{dx},$$

and the variables are separated as

$$\frac{dx}{x} = \frac{dv}{\phi(v) - v}.$$

After performing the integration we replace v by y/x.

Example.—Solve $2xy\dfrac{dy}{dx} = y^2 - x^2$.

We have
$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

The substitution $y = vx$ gives

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

or
$$x\frac{dv}{dx} + \frac{v^2 + 1}{2v} = 0.$$

Hence
$$\frac{2v\,dv}{v^2 + 1} + \frac{dx}{x} = 0,$$

and the variables are separated. Integration gives

$$\log(v^2 + 1) + \log x = \log c$$

or
$$x(v^2 + 1) = c = x\left(\frac{y^2}{x^2} + 1\right).$$

The solution is
$$cx = x^2 + y^2.$$

EXERCISES

1. $x\dfrac{dy}{dx} = y + \sqrt{(x^2 + y^2)}$. [$2cy = c^2x^2 - 1$.]

2. $(x^4 - 2xy^3)\dfrac{dy}{dx} + (y^4 - 2x^3y) = 0$. [$x^3 + y^3 = cxy$.]

3. $x + y\dfrac{dy}{dx} = 2y$. [$\dfrac{c}{y - x} = \exp\dfrac{x}{x - y}$.]

4. $2x^2\dfrac{dy}{dx} = x^2 + y^2$. [$cx = \exp\dfrac{2x}{x - y}$.]

2, 7. Certain non-homogeneous equations can be made homogeneous by a slight modification. The simplest case occurs when the equation has the form

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The "modification" consists in changing the origin so as to eliminate the absolute terms c and C.

Example.—Solve $(10y - x - 21)\dfrac{dy}{dx} = 5x + 4y - 3.$

The two straight lines

$$10y - x - 21 = 0,$$
$$4y + 5x - 3 = 0$$

intersect at $(-1, 2)$. We put the origin there by writing

$$x = X - 1, \quad y = Y + 2.$$

This leaves $dx = dX$, $dy = dY$, so that the equation becomes

$$\frac{dY}{dX} = \frac{5X + 4Y}{10Y - X}. \quad \text{(ii)}$$

We then proceed as before by introducing the new variable v defined by

$$Y = vX, \quad \frac{dY}{dX} = v + X\frac{dv}{dX}.$$

The equation becomes

$$v + X\frac{dv}{dX} = \frac{5 + 4v}{10v - 1},$$

whence

$$X\frac{dv}{dX} = \frac{5 + 4v}{10v - 1} - v = \frac{5 + 5v - 10v^2}{10v - 1}.$$

The variables can now be separated as

$$\frac{5dX}{X} = \frac{10v - 1}{1 + v - 2v^2}dv = \frac{3dv}{1 - v} - \frac{4dv}{1 + 2v}.$$

Integration gives

$$5\log X = -3\log(1 - v) - 2\log(1 + 2v) + \log c.$$

Hence

$$X^5(1 - v)^3(1 + 2v)^2 = c = (X - Y)^3(X + 2Y)^2. \quad \text{(iii)}$$

We now substitute back for the original x and y. The result is

$$(x - y + 3)^3(x + 2y - 3)^2 = c.$$

EXERCISES

1. $\dfrac{dy}{dx} = \dfrac{y - x + 1}{y + x + 5}.$

2. $\dfrac{dy}{dx} = \dfrac{2x + 9y - 20}{6x + 2y - 10}.$

3. Prove that an equation of the form $\dfrac{dy}{dx} = F\left(\dfrac{ax + by + c}{Ax + By + C}\right)$ can be made homogeneous and hence solved by separation.

The above method breaks down if the lines are parallel. The intersection cannot be found if $(ax + by) = m(Ax + By)$ where m is a constant; otherwise expressed, when $a/A = b/B = m$. But in this case the equation 2, 7 (i) becomes

$$\frac{dy}{dx} = \frac{m(Ax + By) + c}{Ax + By + C}.$$

This has the form $$\frac{dy}{dx} = f(Ax + By),$$

which has already been discussed in 2, 4·2. It can be solved by putting $t = Ax + By$.

EXERCISE

Prove that the solution of

$$\frac{dy}{dx} = \frac{3x - 4y - 2}{6x - 8y - 5}$$

is $$\log(6x - 8y - 7) = c - 2x + 4y.$$

[It is suggested that, on first reading, the student should omit the next few sections and proceed to 2, 10.]

2, 8. It is relatively simple to get at the inner meaning of this form. Consider the equation in the form 2, 6 (i). We know that it represents a one-parameter family of curves. Suppose we are supplied with an accurate drawing of this family and we decide to enlarge it. Then x becomes kx and y becomes ky, where k is the scale of enlargement. What is there about the enlargement that would enable us to distinguish it from the original? Nothing whatever. The differential equation of the family remains absolutely unchanged. Remember that all the infinite number of members of the family are present, and though any individual member is moved by enlargement, a smaller one comes up to take its place. This does not apply to all one-parameter families. The enlargement of a family of confocal ellipses, for example, would be detected by the change in the interfocal distance. But if the form is 2, 6 (i) there is no clue to enlargement.

What sort of family would have this property? One of the simplest answers is, a set of lines radiating from the origin. No magnification is going to alter their appearance. Likewise, a set of concentric circles round the origin has the property, and any family that consists of enlargements of a single member, e.g. a set of similar and similarly situated ellipses with centre at the origin.

Let us regard the matter from a different angle. Draw the line $y = mx$ through the origin. It cuts the various members of the family at A_1, A_2, &c. If the tangent at any A make an angle ψ with OX, we have $\tan\psi = dy/dx$. The differential equation 2, 6 (i) then means that $\tan\psi = \phi(m)$, a constant. In other words, the tangents at A_1, A_2, &c., are all parallel. This is only another way of saying that the family consists of similar and similarly situated curves, the origin being the centre of similitude.

In order to test our assertion, consider the family of circles that lie in the first quadrant and touch both co-ordinate axes. No magnification is going to alter their appearance as a family, and so their differential equation should be homogeneous. It has already in I, 7 (ii) been shown to be

$$(x - y)^2(1 + y'^2) = (x + yy')^2.$$

Admittedly this is not of the first degree; but it is homogeneous and can be written

$$(1 - v)^2(1 + y'^2) = (1 + vy')^2,$$

which shows that $\qquad y' = f(v), \quad \text{or} \quad \dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right).$

EXERCISE

Deduce from first principles which of the following one-parameter families would have a homogeneous differential equation:

(i) Circles touching OY at the origin.

(ii) Concentric circles round the point (1, 2) as centre.

(iii) The Boyle's law curve PV = constant.

(iv) Circles of radius unity that pass through the origin.

(v) Parabolas with vertex at the origin touching OY.

Verify your conclusions by finding the differential equations in those cases thought to be homogeneous.

2, 9. From the student's point of view the existence of this homogeneous form is a little unfortunate. No scientist, apart from the mathematician, has ever been able to make use of it. It exercises a curious fascination on examination candidates that is readily explicable. A student, asked to solve a differential equation, is usually faced with an enigma, and there is no general technique of attack. The substitution $y = vx$ offers hope; it only remains to add that usually the student is disappointed. The following is an application to relative velocity.

Example.—A boat which travels at speed u relative to the water sets out from A and makes for B. A current of velocity v runs perpendicular to AB. If the boat is always headed directly for B, find its path. Take B as origin, BA as the x axis, the y axis downstream. Let P be any downstream position of the boat, and suppose PB makes angle θ with BA. The boat then has two velocities; they are v downstream and u along PB. Resolving in the direction of the axes, we get

$$v - u\sin\theta = \dot{y}, \quad u\cos\theta = -\dot{x}.$$

Hence
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{u\sin\theta - v}{u\cos\theta} = \tan\theta - n\sec\theta,$$

if $v = nu$. As $\tan\theta = y/x$, this gives

$$\frac{dy}{dx} = \frac{y}{x} - n\left(1 + \frac{y^2}{x^2}\right)^{\frac{1}{2}},$$

which is homogeneous. The substitution $y = wx$ gives

$$w + x\frac{dw}{dx} = w - n\sqrt{(1 + w^2)}$$

or
$$\frac{dw}{\sqrt{(1 + w^2)}} = -n\frac{dx}{x}.$$

We have, on integration,

$$\sinh^{-1} w = -n\log x + n\log c = \log\left(\frac{c}{x}\right)^n.$$

Hence, since $2\sinh\log\beta = \beta - \beta^{-1}$, we have

$$2w = \left(\frac{c}{x}\right)^n - \left(\frac{x}{c}\right)^n = 2\frac{y}{x}$$

as the equation of the path. It is left to the reader to prove that the transit cannot be effected unless $u > v$.

EXERCISES

1. A moving point has velocities $2x + y - 5$ parallel to OY and $5 - x - 3y$ parallel to OX. Prove that it describes a conic.

2. A point moves in a plane so that its component velocities parallel to the co-ordinate axes are linear functions of the current co-ordinates. Find the condition that it describes a conic.

2, 9·1. A curious point, and one that is rarely mentioned, arises in connexion with the homogeneous equation. If $y = mx$, then $dy/dx = m$, and it follows that $y = mx$ is certainly a solution of 2, 6 (i), provided m is a root of the equation $m = \phi(m)$. This solution, which involves no arbitrary constants, is a particular integral. Taking a previous example, we know that the equation 2, 7 (ii) has the solution 2, 7 (iii). The line $Y = mX$ will be a solution provided that

$$m = \frac{5 + 4m}{10m - 1}, \quad \text{or} \quad 2m^2 - m = 1,$$

whence $m = 1$ or $-\frac{1}{2}$. These evidently correspond to the case where the parameter c is zero. It is left to the reader to prove that in general the solutions obtained as above may be the asymptotes of the family. In particular cases, of course, these may be imaginary. Alternatively, the procedure may give the common tangents at the origin in cases where the whole family pass through the origin.

2, 10. *The Exact Differential.*

Suppose we have $f(x, y)$, a function of x and y which we can equally well denote by z, so that $z = f(x, y)$. For the moment we regard x and y as independent variables and the equation can be visualized as representing a surface. If x receives a small increment dx whilst y remains unchanged, z correspondingly receives a small increment which we denote by dz. Its value, to the first order of small quantities, is $dz = (\partial z/\partial x)\,dx$. Similarly, we might have $dz = (\partial z/\partial y)\,dy$ for a unique change dy in y. If x and y both receive increments, we have

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

This is one of the fundamental theorems in partial differentiation and can be revised in any text on the calculus.

Such a differential, which derives directly from a definite function of x and y, is said to be exact or perfect. For example, if $z = x(x + y)$, then

$$\frac{\partial z}{\partial x} = 2x + y, \quad \frac{\partial z}{\partial y} = x, \quad dz = (2x + y)\,dx + x\,dy.$$

This is an exact differential, and the terminology implies that there are certain differentials which are not exact. This is the case. For example, the differential $(x^2 + y)\,dx + y^2\,dy$ is not exact, in the sense that there is no function of x and y from which it can be derived directly. Such a statement calls for proof, and this we will now obtain by seeking a test which shall decide whether a given differential is exact or not.

Suppose that the differential $M(x, y)\,dx + N(x, y)\,dy$ is exact and has been derived directly from $f(x, y)$: then it is merely another way of writing

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy.$$

Trading on the independence of x and y we can keep y constant, so that dy is zero. This gives at once $M(x, y) = \partial f/\partial x$, and similarly, $N(x, y) = \partial f/\partial y$. Usually $f(x, y)$ is the kind of function in which the order of differentiation is interchangeable, so that

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

This is equivalent to

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

as the test which must be passed if a differential is thought to be exact. Notice that what we have proved is a purely negative statement, that if $\partial M/\partial y \neq \partial N/\partial x$, then $M\,dx + N\,dy$ is not exact. We have *not* proved the proposition in its affirmative form, that if $\partial M/\partial y = \partial N/\partial x$, then $M\,dx + N\,dy$ *is* exact. This happens to be true, but the proof will be deferred; in the meantime we accept it.

When the test is applied to the above differential,

$$(x^2 + y)\,dx + y^2 dy,$$

we have

$$\frac{\partial}{\partial y}(x^2 + y) = 1, \quad \frac{\partial}{\partial x}\,y^2 = 0.$$

The results differ, and thus substantiate our statement that the differential is not exact. Applying our test to the other differential

$$(2x + y)\,dx + x\,dy,$$

we have

$$\frac{\partial}{\partial y}(2x + y) = 1 = \frac{\partial}{\partial x}\,x.$$

The results agree, and the differential passes the test for exactness.

EXERCISES

1. Prove that the following differentials are exact:

(i) $(7x - 5y - 3)dx + (3y - 5x + 7)dy$.

(ii) $(3x^2 + y^2)dx + 2y(x - 1)dy$.

2. Prove that the differential

$$2y(xy - 1)dx - x(xy - 4)dy$$

is not exact, but that it becomes exact if divided by y^3.

2, 11. *Change in Value.*

The simplest way of bringing out the essential difference between an exact differential and one which is not exact is to take specific numerical examples and find the change in value by integrating from one point to another. For simplicity we take the one point to be the origin O and the other to be P with co-ordinates (1, 1). Hitherto we have taken x, y to be independent; but we can make them run in harness by prescribing a path that the point (x, y) shall follow from O to P. The number of such paths is infinite; and here we might as well state in advance what will eventuate. If we work with an exact differential, all routes will give the same result; whereas if the differential is not exact the answer will vary from one path to another.

Drawing the ordinate PQ and the abscissa PR, we have as possible paths from O to P, (i) the straight line OP, (ii) the broken path $OQ + QP$, or $OR + RP$, (iii) the parabolic path $x = y^2$, or $y = x^2$, and so on. We now take the differential $(x^2 + y)dx + y^2dy$ which we know is not exact, and integrate it from O to P along these various routes. We shall find that the answers are all different.

(i) We begin with the straight path OP. Its equation is $y = x$, so that $dy = dx$. Using these values to eliminate y, we have

$$\int_O^P [(x^2 + y)\,dx + y^2\,dy] = \int_O^P [(x^2 + x)\,dx + x^2\,dx]$$

$$= \int_O^P (2x^2 + x)\,dx$$

$$= \Big[\tfrac{2}{3}x^3 + \tfrac{1}{2}x^2\Big]_O^P.$$

On substituting the values of x at O and P, this gives 7/6.

(ii) We now choose the alternative route $OQ + QP$. Along OQ we have $y = 0 = dy$, so that along this part of the path the integrand reduces to $x^2\,dx$. Similarly along PQ we have $x = 1$, $dx = 0$, and the integrand gives $y^2\,dy$. We thus have

$$\int_O^P [(x^2 + y)\,dx + y^2\,dy] = \int_O^Q x^2\,dx + \int_Q^P y^2\,dy$$

$$= \Big[\tfrac{1}{3}x^3\Big]_0^1 + \Big[\tfrac{1}{3}y^3\Big]_0^1 = 2/3.$$

The reader can similarly prove for himself that the route $OR + RP$ would give 5/3.

(iii) As a last path we take the parabolic route $y = x^2$, so that $dy = 2x\,dx$. This gives

$$\int_O^P [(x^2 + y)\,dx + y^2\,dy] = \int_O^P [(x^2 + x^2)\,dx + x^4 \,.\, 2x\,dx]$$

$$= \int_O^P (2x^2 + 2x^5)\,dx$$

$$= \Big[\tfrac{2}{3}x^3 + \tfrac{1}{3}x^6\Big]_0^1 = 1.$$

This is characteristic of differentials which are not exact; the integrated value between two points is not unique but depends upon the path chosen. A different state of affairs prevails when we work with a differential which is known to be exact; we find that all routes give the same result. We can take our previous example $(2x + y)\,dx + x\,dy$ which derives from $x(x + y)$.

(i) The straight path OP gives

$$\int_O^P [(2x + y)\,dx + x\,dy] = \int_O^P 4x\,dx = \Big[2x^2\Big]_0^1 = 2.$$

All other paths will give this same value.

(ii) The parabolic path $y = x^2$ gives

$$\int_O^P [(2x + y)\,dx + x\,dy] = \int_O^P [(2x + x^2)\,dx + x \,.\, 2x\,dx]$$

$$= \int_O^P (2x + 3x^2)\,dx$$

$$= \Big[x^2 + x^3\Big]_0^1 = 2.$$

It is hardly worth while pursuing the matter further, for since

$$d[x(x + y)] = (2x + y)\,dx + x\,dy$$

we must have

$$\int_O^P [(2x + y)\,dx + x\,dy] = \int_O^P d[x(x + y)]$$
$$= \Big[x(x + y)\Big]_O^P = 2,$$

a result which is unique and independent of the path.

EXERCISES

1. The differential $3y\,dy + 4x(x\,dy + y\,dx)$ is not exact. Compute its integral from (0, 0) to (1, 2) along any two paths, one of which is the parabola $y = 2x^2$.

2. The differential $y(2 - y)\,dx + x(1 - y)\,dy$ is not exact. Calculate its integral between (0, 0) and (2, 1) along any two broken linear paths and along the direct path.

These numerical results hardly constitute a proof. They serve as illustrations of the essential difference between exact differentials and those which are not exact. They may serve to convince those who do not wish to delve deeper into the matter that our statements do at least possess an air of verisimilitude. One part admits of simple direct proof; for if $M\,dx + N\,dy$ is known to derive directly from $f(x, y)$ and we decide to integrate from the origin to some point A with co-ordinates (a, b), we have $M\,dx + N\,dy = df$, so that

$$\int_O^A (M\,dx + N\,dy) = \int_O^A df = \Big[f(x, y)\Big]_O^A = f(a, b) - f(0, 0),$$

a result which depends solely on the position of A and nowise on the route followed. The distinction between exact differentials and those which are not exact is of considerable importance in thermodynamics. It forms the basis of the proof that the entropy is a function of the physical co-ordinates and leads to Maxwell's thermodynamic equations.

2, 12. *Green's Lemma.*

In certain branches of mathematical physics, such as electrostatics and hydrodynamics, considerable use is made of certain theorems which convert triple or volume integrals to double or surface integrals, and double integrals to single or line integrals. One of the simplest of these theorems, really a degenerate two-dimensional form of Stokes's more general three-dimensional theorem, is sometimes known as Green's lemma for the plane.

Suppose we have a simple closed plane curve, simple in the sense that it does not cross itself, so that it may be visualized as a sort of oval. If M and N be two functions of x and y, the theorem states that a certain double integral, taken over the area of the curve, is equivalent to a single integral taken round the boundary. The theorem takes the following form:

$$\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy=\int(M\,dx+N\,dy),$$

and the conditions under which it is valid are established in works on the integral calculus.

If we accept this proposition we get a new aspect of our problem. Take two points A, B and join them by a number of paths APB, AQB, ARB, &c., which nowhere cross themselves or each other; then any two of them will serve as an area and its boundary. If the integrand in the single integral on the right is an exact differential, the integrand on the left is zero by the test. In this case both integrals have zero value. The single integral is supposed to be taken right round the boundary such as $APBQA$. If we take it in two parts, we have, in this case,

$$\int(\text{along } APB)+\int(\text{along } BQA)=0.$$

Reversing along the second part, we can write

$$\int(\text{along } APB)-\int(\text{along } AQB)=0,$$

or

$$\int(\text{along } APB)=\int(\text{along } AQB).$$

But as the contour might equally well have been $APBRA$, each of the last two integrals is equal to $\int(\text{along } ARB)$. In other words, the integral from A to B is independent of the path if $M\,dx+N\,dy$ is exact.

We now come to the alternative case where the integrand of the double integral is not zero, so that the integrand on the right is not exact. It might happen fortuitously that the double integral, with its non-zero integrand, gave zero when applied to some particular area; but this would be the merest chance. It could not give zero when applied to any and every area, and hence the single integral is not usually zero. This means that the integral from A to B varies with the route, the previous equalities being replaced by inequalities.

2, 13. *Method of Integration.*

The two points that remain to be cleared up are, (i) the proof of the converse, that if $\partial N/\partial x = \partial M/\partial y$, then the differential is exact; (ii) the determination of the method of integration when the test is satisfied. The first is contained in the second. Assuming then that $M\,dx + N\,dy$ is exact, we desire to find the $f(x, y)$ from which it derives. Marking a fixed starting-point A with co-ordinates (a, b), and a variable point P with co-ordinates (x, y), we know that the integral from A to P is independent of the route. Naturally we pick the easiest route. We accordingly draw the vertical through A and the horizontal through P to intersect at D with co-ordinates (a, y), and we choose the broken path $AD + DP$. Along AD we have $x = a$, $dx = 0$, so that the differential $M(x, y)\,dx + N(x, y)\,dy$ becomes $N(a, y)\,dy$. Along DP we have $dy = 0$, and the differential there becomes $M(x, y)\,dx$. Hence

$$\int_A^P (M\,dx + N\,dy) = \int_a^x M(x, y)\,dx + \int_b^y N(a, y)\,dy.$$

The result will be some function of x and y that we can denote by $f(x, y)$. Had another starting-point B been chosen instead of A, the results would differ by the integral from B to A. With A and B fixed, this difference is a definite constant, and as we expect an arbitrary constant anyway in the indefinite integral (indefinite since P is a variable point), we conclude that the starting-point A is immaterial. In practice we ignore completely the lower limit b, which merely contributes a constant, and give a any value that suits us, usually 1 or 0. The following example shows the method.

Example.—Integrate $(x^2 + \log y)\,dx + \dfrac{x}{y}\,dy$.

The test is satisfied since

$$\frac{\partial}{\partial y}(x^2 + \log y) = \frac{1}{y} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right).$$

A suitable value for a is 0, and we write

$$\int\left[(x^2 + \log y)\,dx + \frac{x}{y}\,dy\right] = \int_0^x (x^2 + \log y)\,dx + \int^y \frac{0}{y}\,dy + c$$
$$= \tfrac{1}{3}x^3 + x\log y + c.$$

EXERCISE

Integrate the three examples at the end of 2, 10, which are known to be exact.

We can now clear up the former of the above two points. With

$$f(x, y) = \int_a^x M(x, y)\,dx + \int_b^y N(a, y)\,dy,$$

we have
$$\frac{\partial f}{\partial x} = M(x, y),$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_a^x M(x, y)\,dx + N(a, y)$$

$$= \int_a^x \frac{\partial}{\partial y} M\,dx + N(a, y).$$

In virtue of the test, this can be written

$$\int_a^x \frac{\partial N}{\partial x}\,dx + N(a, y) = \Big[N(x, y)\Big]_a^x + N(a, y) \equiv N(x, y).$$

Hence
$$M\,dx + N\,dy = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy,$$

which is exact.

2, 14. *The Exact Equation.*

It was deemed necessary to spend so much time in emphasizing the real nature of the distinction between exact and non-exact differentials because experience shows that it is just as rarely appreciated as the method of integration is understood. The reason for this lacuna is not far to seek. The matter is of no great importance, until one begins to study a subject like thermodynamics, when it suddenly jumps into the front rank.

Turning now from differentials to differential equations, the first-order equation when written in the form $M\,dx + N\,dy = 0$ is said to be exact if the left side is an exact differential. We now know that a necessary condition for exactness is that $\partial M/\partial y = \partial N/\partial x$, in the sense that unless this condition is fulfilled the differential cannot be exact. We also know that the condition is of the type known as "sufficient", in the sense that if the condition is fulfilled, the differential certainly is exact. An alternative proof of sufficiency will be found in Piaggio's *Differential Equations* or Lamb's *Infinitesimal Calculus*.

If the left side is equivalent to df, where $f = f(x, y)$, then the equation is equivalent to $df = 0$, and has the solution $f(x, y) = c$. Hence it is sometimes stated that exact equations derive directly from equations of the type $f(x, y) = c$ without the removal of factors. It is perhaps as well to add that when the exact equation occurs in practice, its form is usually so simple that one can see at sight whence it derives.

2, 15. *The Integrating Factor.*

The chief interest in the exact equation is theoretical. A previous exercise in 2, 10 suggests that when the equation is not exact it may become exact on inserting a factor. This is invariably the case, so that all first-order first-degree equations come under the "exact" type. The factor in question is known as an "integrating factor". Assuming $\partial M/\partial y \neq \partial N/\partial x$, let us multiply the equation by $J(x, y)$ with the intention of making it exact. The equation now reads

$$J(x, y)M(x, y)\,dx + J(x, y)N(x, y)\,dy = 0,$$

and the test for exactness gives

$$J\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial J}{\partial x} - M\frac{\partial J}{\partial y}.$$

This equation serves to determine J. Unfortunately its determination is usually at least as difficult as solving the original equation by other means. All the same there is a certain value in knowing that J exists. Matters are occasionally simplified by J turning out to be a function of one variable only. In any case the number of integrating factors is infinite; for if a knowledge of J reduces the original equation to $du = 0$, whence $u(x, y) = c$, then the factor $Jf'(u)$ would reduce it to $f'(u)\,du = 0$, whence $f(u) = c$, which is equivalent to $u(x, y) = c$.

Example.—The equation $4xy\,dx + (4x^2 + 3y)\,dy = 0$ is not exact. If we multiply by J we can write

$$4Jxy\,dx + J(4x^2 + 3y)\,dy = 0.$$

This is exact if

$$\frac{\partial}{\partial y}(4Jxy) = \frac{\partial}{\partial x}[J(4x^2 + 3y)].$$

Therefore
$$4Jx + 4xy\frac{\partial J}{\partial y} = 8Jx + (4x^2 + 3y)\frac{\partial J}{\partial x}.$$

If J is a function of y alone, we have $\partial J/\partial x = 0$, and the equation reduces to

$$y\frac{\partial J}{\partial y} = J.$$

On separation of the variables we reach $J = y$ as the simplest solution. Multiplication by this factor converts the original equation to

$$4xy^2\,dx + y(4x^2 + 3y)\,dy = 0 = du.$$

If it is not evident on sight that $u = 2x^2y^2 + y^3$, we proceed in the usual manner indicated previously. Choosing $x = 0$ as a suitable starting value, we have

$$\int_0^x 4xy^2\,dx + \int^y 3y^2\,dy = 2x^2y^2 + y^3,$$

and the solution is $2x^2y^2 + y^3 = c$. The integrating factor might equally well have been $yf'(2x^2y^2 + y^3)$, where f is any function whatever. The solution would then have run

$$f(2x^2y^2 + y^3) = c,$$

which is equivalent to the previous result.

EXERCISES

1. The equation $2\tan x \tan y\,dx + \sec^2 y\,dy = 0$ is evidently not exact. Find an integrating factor which is a function of x, and thus integrate the equation. Verify the result by separating the variables.

2. Verify a previous statement in 2, 10 that y^{-3} is an integrating factor for the equation $2y(xy - 1)\,dx - x(xy - 4)\,dy = 0$.

3. Integrate the equation $y(2 - y)\,dx + x(1 - y)\,dy = 0$ on the assumption that it has an integrating factor which is a function of x.

4. The differential of a function of x and y was correctly computed as

$$(2x\cos y - y^3\sec^2 x)\,dx - (\qquad)\,dy.$$

Unfortunately, the terms in the second bracket were obliterated. What was the function from which it derived?

5. Prove that the integrating factor J necessarily exists as a function of x if $\dfrac{1}{N}\left\{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right\}$ is independent of y. If this expression has the value $F(x)$, prove that $J = \exp(\int F\,dx)$. Apply the method to the first and third exercises above.

6. Prove the following rule, sometimes given in text-books, for integrating $M\,dx + N\,dy$, when it is known to be exact and M, N are polynomials:

(i) Integrate M with respect to x; (ii) Integrate the terms in N which do not contain x; (iii) Equate the sum to a constant.

7. Verify that the non-exact equation $\dfrac{dy}{dx} = \dfrac{y(xy + 1)}{x(xy - 1)}$ has the integrating factor $(xy)^{-2}$ and hence deduce the solution $\log\dfrac{x}{y} = c + \dfrac{1}{xy}$.

2, 16. *Applications.*

The idea of the exact differential plays a considerable part in applied mathematics. As a first illustration we may consider plane conservative fields of force.

Suppose a force F has components X and Y. Let the point of application have an arbitrary displacment ds whose components are dx and dy. Then dW, the element of work done, is given by

$$dW = X\,dx + Y\,dy.$$

In certain cases, depending on the nature of F, the element dW is an exact differential of the co-ordinates x and y. In these cases it is customary to replace dW by $-dV$ and V is called the potential. The above equation then reads

$$-dV = X\,dx + Y\,dy,$$

from which we deduce the two equations

$$X = -\frac{\partial V}{\partial x};\ Y = -\frac{\partial V}{\partial y}. \quad \text{(i)}$$

These state that the force in any direction is the negative gradient of the potential in that direction.

The work done in a finite displacement from A to B is

$$\int_A^B (X\,dx + Y\,dy) = -\int_A^B dV = V_A - V_B.$$

Expressed in words, the work done is equal to the potential drop from A to B. It is independent of the route from A to B, so that the work done in passing from A to B and back by a different route to A is zero. Such a field of force is said to be conservative. The loss of potential is shown by a corresponding increase in the kinetic energy of the system.

The two equations (i) imply that X and Y are themselves functions of x and y. It follows from this that a field cannot be conservative if friction is present. The complete specification of the frictional force demands a knowledge of its direction of action. As this is opposite to the direction of motion it cannot be specified from a mere knowledge of the position.

EXERCISE

An elastic string, of natural length c, has one end fixed and is stretched to a length λ. The other end is attached to a mass m. If k be the force required to give unit extension to the string, and the motion is uniplanar without friction, prove that the system is conservative.

As a second illustration we may take the two-dimensional flow of an incompressible fluid, the motion being the same in all horizontal planes. If a particle of the fluid has component velocities u and v, its displacements dx and dy in time dt are governed by the stream-line relation

$$\frac{dx}{u} = \frac{dy}{v}, \text{ or } v\,dx - u\,dy = 0. \quad \ldots \quad \text{(ii)}$$

Cognizance is taken of the incompressibility of the fluid by the so-called "equation of continuity", which in the language of vectors means that the vector velocity has zero divergence. In two dimensions this takes the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad \ldots \quad \text{(iii)}$$

This result shows that the differential (ii) is exact. It is customary to denote it by $d\psi$, so that

$$d\psi = v\,dx - u\,dy,$$

whence

$$u = -\frac{\partial \psi}{\partial y};\ v = \frac{\partial \psi}{\partial x}.$$

The stream-lines are then the one-parameter family of curves $\psi = c$. The function ψ is known as the stream-function or current-function. It has the property that the change in value from a point A to another point B measures the flow across any line joining A and B.

The stream-line equation (ii) can be written

$$\frac{dy}{dx} = \frac{v}{u}.$$

If there exists a family of curves orthogonal to these, its equation would be

$$\frac{dy}{dx} = -\frac{u}{v}, \text{ or } u\,dx + v\,dy = 0. \quad \ldots \quad \text{(iv)}$$

If this happens to be an exact differential $-d\phi$, the component velocities are given by

$$u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y}.$$

Borrowing the idea of the force-potential, this function ϕ is known as the velocity potential; the equation of continuity (iii) shows that it satisfies Laplace's equation in two dimensions,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

When the differential (iv) is exact, so that ϕ exists, the motion is said to be irrotational. Alternatively, when (iv) is not exact, the use of an integrating factor may still lead to a family of curves orthogonal to the stream-lines; but the velocity potential no longer exists, and the motion is distinctively called rotational or vortex.

EXERCISE

If the velocities are given by $u = -cy$, $v = cx$, verify that the equation of continuity is satisfied. Prove that the stream-lines are the circles $x^2 + y^2 =$ const.; but that the motion is vortical, as no velocity potential exists.

As a third illustration we may consider the thermodynamics of a simple gas. The state of a unit mass is taken to be defined by its pressure P, its volume V, and the absolute temperature T. These three co-ordinates are not independent; they are connected by a single relation which we may denote by $f(P, V, T) = 0$. It is known as the "equation of state", or the "characteristic equation". Incidentally it is no easy matter to find a characteristic equation that is valid over a wide range; but several examples are in common use over restricted ranges of temperature and pressure. The existence of the characteristic equation means that only two of the variables are independent.

In accordance with the conservation of energy it is taken that the addition of an element of heat dQ to the unit mass of gas may increase the internal energy E or cause work to be done against the external pressure, so that

$$dQ = dE + dW.$$

Concerning E, no assumption need be made beyond its being a function of the co-ordinates. As for dW, it has the form $P\,dV$ for a gas under uniform external pressure, so that

$$dQ = dE + P\,dV. \qquad \text{(v)}$$

The right side is now a function of the co-ordinates; nevertheless, the differential dQ is not exact. The reason is that if the gas be changed from a state A to a state B, the heat-addition (or subtraction) will depend on the route followed from A to B; for this may be constructed from endless combinations of changes under constant pressure, constant volume, isothermal or adiabatic.

In virtue of the characteristic equation it is possible to eliminate any one of the variables from (v), so that the equation is capable of taking the three other forms

$$\left.\begin{aligned} dQ &= l_1 dV + k_1 dP \\ &= l_2 dV + k_2 dT \\ &= l_3 dP + k_3 dT \end{aligned}\right\} . \qquad \text{(vi)}$$

All that is known for the moment about the l's and k's is that they are functions of the co-ordinates, not necessarily constants; nevertheless, it is possible to attribute physical meanings to them. Thus, if we take the volume constant in the second relation, we have

$$dV = 0, \quad k_2 = \left(\frac{\partial Q}{\partial T}\right)_V,$$

showing that k_2 is the rate of addition of heat with temperature, i.e. the specific heat at constant volume. Similarly,

$$l_3 = \left(\frac{\partial Q}{\partial P}\right)_T,$$

and this, being a heat change at constant temperature, must be some sort of latent heat.

We can now introduce the idea of "entropy". If the addition of a heat-element dQ causes a reversible change from a position of thermal and mechanical equilibrium, we take another variable ϕ which is known as the entropy, and is defined by the differential relation $d\phi = dQ/T$. One immediate consequence from this definition relates to adiabatic changes. Here dQ is necessarily zero, so that ϕ is constant. For this reason adiabatics are sometimes called isentropics.

It is a logical consequence from the theory of Carnot's cycle that the change of entropy round any reversible cycle is zero. If A and B are two positions on a reversible cycle, the value of $\phi_B - \phi_A$ is unaffected by the direction in which the cycle is performed. Moreover, A and B can be connected by any number of reversible cycles, and if A be taken as a standard position or origin, the entropy at B depends solely on B's position. This is only another way of saying that ϕ is a function of the co-ordinates and $d\phi$ is an exact differential.

The conservation of energy (v) can now be written

$$dE = T\,d\phi - P\,dV.$$

In addition to E and ϕ it is customary to use certain other thermodynamic functions. Thus the addition of

$$d(PV) = P\,dV + V\,dP$$

leads to

$$d(E + PV) = T\,d\phi + V\,dP.$$

Similarly, the subtraction of

$$d(T\phi) = T\,d\phi + \phi\,dT$$

gives

$$d(E - T\phi) = -P\,dV - \phi\,dT.$$

If both operations be performed, we have

$$d(E + PV - T\phi) = V\,dP - \phi\,dT.$$

The function $E + PV$ is called the "total heat" by British engineers, but there is a tendency to use the name "enthalpy". The function $E - T\phi$ is the "free energy", whilst $E + PV - T\phi$ is known as Gibbs' function G. It has the useful property of being constant for isothermal changes in a saturated vapour, since dT and dP are then both zero. All these are functions of the co-ordinates, and their differentials are accordingly exact. On applying the usual test we reach in turn the four important thermodynamic relations due to Clerk Maxwell:

(1) $\left(\frac{\partial T}{\partial V}\right)_\phi = -\left(\frac{\partial P}{\partial \phi}\right)_V$, (2) $\left(\frac{\partial T}{\partial P}\right)_\phi = \left(\frac{\partial V}{\partial \phi}\right)_P$,

(3) $\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial \phi}{\partial V}\right)_T$, (4) $\left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial \phi}{\partial P}\right)_T$.

Any good set of physical tables must satisfy these relations. They are easily remembered by the following three rules:

(a) In every case they cross-multiply to PV or $T\phi$, both of which have the dimensions of work.
(b) The subscripts are the denominators interchanged.
(c) The minus occurs when T and V are in the same fraction.

Reverting to the second of the relations (vi) we have already seen that k_2 is really k_v; moreover, the definition $dQ = T\,d\phi$ shows that $1/T$ is the integrating factor that makes the right side exact, and we can write

$$d\phi = \frac{l_2}{T}\,dV + \frac{k_v}{T}\,dT.$$

By keeping T constant and $dT = 0$, we derive the value of l_2 as

$$l_2 = T\left(\frac{\partial \phi}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V,$$

the last step being in virtue of the third Maxwell relation. We accordingly have

$$T\,d\phi = T\left(\frac{\partial P}{\partial T}\right)_V dV + k_v dT$$

for any change that may occur among the variables. If the change takes place at a constant pressure, we have, on division by dT,

$$T\left(\frac{\partial \phi}{\partial T}\right)_P = T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P + k_V.$$

The left side is equivalent to $(\partial Q/\partial T)_p$ and is therefore k_p. We thus reach a relation for the difference of the specific heats

$$k_p - k_V = T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P,$$

which is calculable from the characteristic equation.

EXERCISES

1. Taking $PV = RT$ as the characteristic equation of a perfect gas, prove that the difference of the specific heats is R. Further, assuming that k_V is constant, show that the entropy change is given by

$$\varphi_2 - \varphi_1 = k_V \log \frac{T_2}{T_1} + R \log \frac{v_2}{v_1}.$$

2. Prove that, from a suitable set of tables, the entropy is calculable as

$$\varphi = -\left(\frac{\partial G}{\partial T}\right)_P.$$

3. Prove that the enthalpy H is connected with the Gibbs function G by the relation

$$G - H = T\left(\frac{\partial G}{\partial T}\right)_P.$$

4. A chart shows adiabatics and isothermals with P plotted vertically and V horizontally. Prove that at any point the ratio of the specific heats is given by k_P/k_V = (slope of adiabatic)/(slope of isothermal). As the adiabatics are steeper we conclude that k_P exceeds k_V.

5. It is a property of partial differential coefficients that if $f(x, y, z) = 0$, then

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x = -1.$$

Deduce that the difference of the specific heats is given by

$$k_V - k_P = T\left(\frac{\partial V}{\partial T}\right)_P^2 \left(\frac{\partial P}{\partial V}\right)_T.$$

Hence show that k_P must exceed k_V for any substance whose volume reduces with increasing pressure at constant temperature.

2, 17. *The Linear Equation.*

The linear equation of the first order has the form

$$\frac{dy}{dx} + Py = Q, \quad \ldots \ldots \ldots \quad \text{(i)}$$

where P, Q are in general functions of x which in particular cases may degenerate to mere constants, including zero. Note that if P and Q are both constants the variables are separable; and the same is true if either is zero.

We begin by making an observation that applies to all linear equations, whether of the first order or higher. If we substitute $y = u + v$, we have

$$\frac{du}{dx} + Pu + \frac{dv}{dx} + Pv = Q,$$

and this is satisfied provided

$$\frac{dv}{dx} + Pv = Q,$$

$$\frac{du}{dx} + Pu = 0. \quad \ldots \ldots \ldots \quad \text{(ii)}$$

It appears from this that the solution of the original equation falls into two parts, u and v. For the purpose of distinction, u is called the "complementary function". It will be noticed that it satisfies the "reduced equation", the original equation when Q is replaced by zero. It will eventuate in a later chapter that there is a special technique for finding u, and it supplies all the arbitrary constants that the equation requires. In the present case, working with an equation of the first order, only one such constant is required. If we separate the variables in (ii), we have

$$\frac{du}{u} + P\,dx = 0.$$

Integration gives $\quad \log u + \int P\,dx = \log c,$

whence $\quad u = c \exp(-\int P\,dx).$

This is the complementary function containing one constant.

Reverting to v, the other part of the solution, we note that it satisfies the equation as it stands. It is called the "particular integral", and in practice it is taken to be any solution whatever, the simpler the better. It may be obtained by trial or guesswork; but in all the commoner cases there is a specific technique for its determination.

2, 18. *Methods of Solution.*

Before proceeding to the usual method of solution it is worth while to look at the method employed by Bernoulli; in fact, the first order linear equation is frequently referred to as the Bernoulli type. He assumed that y was the product of two functions of x, an assumption possessing the advantage that one of the functions can be made to satisfy any convenient condition. With $y = zw$ and

$$\frac{dy}{dx} = z\frac{dw}{dx} + w\frac{dz}{dx},$$

the linear equation becomes

$$z\left(\frac{dw}{dx} + Pw\right) + \left(w\frac{dz}{dx} - Q\right) = 0.$$

If we equate one of these brackets to zero, the other automatically equals zero. We accordingly have from the first bracket

$$w = c \exp(-\int P\,dx),$$

whence the second bracket gives

$$z = \int \frac{Q}{w}\,dx.$$

For given values of P and Q, the values of w and z (and hence y) are all determinable.

Example.—Solve the equation $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2.$

This is equivalent to

$$z\left(\frac{dw}{dx}+\frac{w}{x}\right)+\left(w\frac{dz}{dx}-x^2\right)=0.$$

Hence $w = a/x$ and

$$\frac{dz}{dx}=\frac{x^2}{w}=\frac{x^3}{a},$$

whence $z = x^4/4a + b$, giving

$$y = wz = \frac{x^3}{4}+\frac{c}{x},$$

on replacing ab by c.

The usual method of solution is to seek an integrating factor $J(x)$ with the hope of making the left side an exact differential. The equation 2, 17 (i) becomes

$$J\,dy + JPy\,dx = JQ\,dx. \quad \text{(i)}$$

The test for exactness, when applied to the left side, gives (since neither J nor P contains y)

$$\frac{\partial J}{\partial x}=\frac{\partial}{\partial y}(JPy)=JP,$$

or

$$dJ = JP\,dx. \quad \text{(ii)}$$

This certainly makes the left side exact, and we can now write (i) as

$$J\,dy + y\,dJ = JQ\,dx = d(Jy).$$

Hence

$$Jy = \int JQ\,dx + c.$$

The factor J is obtained from (ii) by separation of the variables. In its simplest form it is $J = \exp\int P\,dx$. The rule therefore is, first determine the integrating factor from $J = \exp\int P\,dx$, and then complete the solution from

$$Jy = \int JQ\,dx + c.$$

Example.—Our previous example $\frac{dy}{dx}+\frac{y}{x}=x^2$ gives

$$J = \exp\int x^{-1}\,dx = \exp\log x = x.$$

Hence

$$xy = \int x^3\,dx + c = \frac{x^4}{4}+c,$$

in agreement with the previous result.

It is perhaps as well to point out that the equation must first be thrown into the correct form.

Example.—Solve $x\cos x\dfrac{dy}{dx}+y(x\sin x+\cos x)=1$.

We re-write as $$\frac{dy}{dx}+y(\tan x+x^{-1})=x^{-1}\sec x.$$

Here $$J=\exp[\int(\tan x+x^{-1})dx]=\exp(\log\sec x+\log x)=x\sec x.$$

This gives $$xy\sec x=\int\sec^2 x\,dx+c=\tan x+c.$$

2, 19. *Application.*

The linear equation is of very common occurrence and has a wide range of applications.

Example.—An alternating voltage $E_0\sin\omega t$ acts on a circuit of negligible capacity with resistance R and inductance L. The application of Ohm's law gives for the current I the equation

$$RI=E_0\sin\omega t-L\frac{dI}{dt}.$$

Writing $$\frac{dI}{dt}+aI=b\sin\omega t,\ a=\frac{R}{L},\ b=\frac{E_0}{L},$$

we have $$J=\exp\int a\,dt=e^{at},$$

whence $$Ie^{at}=b\int e^{at}\sin\omega t\,dt+c$$

$$=\frac{be^{at}\sin(\omega t-\varphi)}{\sqrt{(a^2+\omega^2)}}+c,\ \tan\varphi=\frac{\omega}{a}=\frac{\omega L}{R}.$$

Therefore $$I=c\exp\left(\frac{-R}{L}t\right)+\frac{E_0\sin(\omega t-\varphi)}{\sqrt{(R^2+\omega^2L^2)}}.$$

The first term here is the complementary function. As so often happens it is a negative exponential. It accordingly decays very rapidly with the passage of time and soon ceases to be of any consequence; hence it is known technically as a transient. The angle φ is the phase, and since it is here subtracted the current is said to lag. The square root in the denominator has the dimensions of R, and is obviously greater than R. It is known as the *impedance*, and the presence of L shows that the effect of the inductance is to diminish the amplitude of the current.

2, 20. *Bernoulli's Equation.*

The solution of differential equations frequently calls for a display of ingenuity. One lays down rules and sets up standard forms of attack; but it is not always easy to decide under what form a particular equation comes, or whether it comes under any. A change of dependent variable is sometimes effective; or it may be necessary to interchange the roles of the variables. Occasionally more than one method is valid for the same equation, but no rules can be laid down to cover all cases, and facility comes only with practice and experience.

The commonest variation on the first-order linear form is known as Bernoulli's equation, which is the non-linear equation

$$\frac{dy}{dx} + Py = Qy^n.$$

It can be written as
$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q.$$

Choosing a new dependent variable $u = y^{1-n}$, we have

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

Hence
$$\frac{1}{1-n}\frac{du}{dx} + Pu = Q,$$

which is linear.

Example.—Consider the equation $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2$,

which can be written
$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1.$$

Put
$$u = \frac{1}{y}, \quad \frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}.$$

The equation becomes
$$-\frac{du}{dx} + \frac{u}{x} = 1.$$

It is easily verified that this has $J = 1/x$, and leads to $u/x = \log(c/x)$. Hence the solution is

$$xy\log\left(\frac{c}{x}\right) = 1.$$

2, 20·1. Although the procedure for solving the linear equation is quite straightforward and reducible to rule of thumb, it is not always possible to achieve a result in finite form. This is owing to the difficulties of performing the integrations; but the result none the less counts as a solution. Consider a series circuit of negligible capacity, containing a battery of constant E.M.F. $= E$. Let L be the inductance, and suppose the resistance varies sinusoidally with the time, so that $R = A\sin\omega t$. The equation for the current is

$$RI = E - L\frac{dI}{dt},$$

or
$$\frac{dI}{dt} + \frac{IA}{L}\sin\omega t = \frac{E}{L}.$$

The integrating factor is

$$J = \exp\int\frac{A}{L}\sin\omega t\,dt = \exp\Big[-\frac{A}{\omega L}\cos\omega t\Big].$$

The "solution" is thus

$$I\exp\Big[-\frac{A}{\omega L}\cos\omega t\Big] = \int\frac{E}{L}\exp\Big[-\frac{A}{\omega L}\cos\omega t\Big]dt + B.$$

The integral on the right is not expressible in finite terms, and in default of proceeding to very much higher mathematics the result would be left as it stands, for what it is worth. Frequently no further progress can be made in such cases; actually the result here can be expressed in an infinite series of Bessel functions, and the example is a first approximation to the problem of the variable resistance microphone.

EXERCISES, 2, 21

1. $(x^2 - 1)\dfrac{dy}{dx} = x(y - a).$ $[y = a + c\sqrt{(1 - x^2)}.]$

2. $y + \dfrac{dy}{dx}\cos^2 x = c\tan x.$ $[y = c(\tan x - 1) + a\exp(-\tan x).]$

3. $\dfrac{dy}{dx}\cos x = y\sin x - \sin 2x.$ $[y = \cos x + c\sec x.]$

4. $x(x^2 - 1)\dfrac{dy}{dx} = y(3x^2 - 2) + x^4.$ $[y = cx^2\sqrt{(x^2 - 1)} - x^2.]$

5. $\dfrac{dr}{d\theta}\sin\theta - r\cos\theta = Ar^3.$ $[\sin^2\theta = r^2(c + 2A\cos\theta).]$

6. $(x + 2y^3)\dfrac{dy}{dx} = y.$ $[x = y^3 + cy.]$

7. $\dfrac{dy}{dx}\cos x + y^2 = y\sin x.$ $[y = 1/(\sin x + c\cos x).]$

8. Find a curve which satisfies the equation $\dfrac{dy}{dx} + y\tan x = \sec x$, and makes an intercept of two on OY. $[y = \sin x + 2\cos x.]$

9. Solve the equation $\dfrac{dy}{dx} + y\cos x = \cos x$. See 2, 5, Ex. 10.

MISCELLANEOUS EXERCISES ON CHAPTER II

1. $(1 + x^2)\dfrac{dy}{dx} = 1 + xy.$ $\qquad [y = x + c\sqrt{(1 + x^2)}.]$

2. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}.$ $\qquad [\log(x^2 + y^2) + 2\tan^{-1} y/x = c.]$

3. $\dfrac{dy}{dx} = \dfrac{y^2(x + 1)}{x^2(y - 1)}.$ $\qquad \left[\log\dfrac{x}{y} = \dfrac{1}{x} + \dfrac{1}{y} + c.\right]$

4. $xy\dfrac{dy}{dx} = x + y^2.$ $\qquad [2x + y^2 = cx^2.]$

5. $\dfrac{dy}{dx} = \dfrac{y}{x - y}.$ $\qquad [y\exp(x/y) = c.]$

6. $(1 + x)y\,dx + (1 - y)x\,dy = 0.$ $\qquad [y + c = x + \log(xy).]$

7. $\dfrac{y}{x} = \dfrac{dy}{dx} + x.$ $\qquad [x^2 + y = cx.]$

8. $x\dfrac{dy}{dx} + y = y^2\log x.$ $\qquad [y(1 + \log x + cx) = 1.]$

9. $y\dfrac{dy}{dx} + y^2 = \cos x.$ $\qquad [2\sin x + 4\cos x + ce^{-2x} = 5y^2.]$

10. $(2x^2 + 4xy - y^2)\dfrac{dy}{dx} = x^2 - 4xy - 2y^2,$
which is both homogeneous and exact. $\qquad [x^3 + y^3 = c + 6xy(x + y).]$

11. $yy' = x(yy'' + y'^2).$
(See I, 7.) $\qquad [y^2 = ax^2 + b.]$

12. $2y'y''' = 3y''^2.$
(See I, 8, Ex. 10.)

13. $xyy'' - xy'^2 + yy' = 0.$ $\qquad [y = ax^b.]$

CHAPTER III

The Linear Equations; Constant Coefficients

3, 1. *The Linear Equation above the First Order.*

We now turn to the linear equation of order higher than the first. We confine our attention mainly to the case where the coefficients are constants, and this for two reasons. It happens to be particularly important in science; and when the coefficients are not constants we touch the higher realms of our subject.

The linear equation with constant coefficients has the form

$$ay + b\frac{dy}{dx} + c\frac{d^2y}{dx^2} + \ldots = f(x). \quad . \quad . \quad . \quad . \quad \text{(i)}$$

We begin by exemplifying a statement made previously. Assuming that $y = u + v$, we have

$$(au + bu' + cu'' + \ldots) + (av + bv' + cv'' + \ldots) = f(x).$$

This is satisfied provided

$$av + bv' + cv'' + \ldots = f(x),$$

and

$$au + bu' + cu'' + \ldots = 0.$$

Here u is the complementary function, derived from what is known as the reduced or auxiliary equation, formed by assuming that the right-hand side is zero. The other term v is any solution of the equation as it stands, and is the particular integral. The full solution is $y = u + v$.

The auxiliary equation has the useful property that its solutions are additive, so that if $u_1, u_2, \ldots$ are solutions, so also is $Au_1 + Bu_2 + \ldots$ where $A, B, \ldots$ are any arbitrary constants. The proof is simple. If u_1 is a solution, then

$$au_1 + bu_1' + cu_1'' + \ldots = 0.$$

Similarly,

$$au_2 + bu_2' + cu_2'' + \ldots = 0,$$

and so on. On multiplying by any arbitrary constants A, B, &c., and adding, we have

$$aU + bU' + cU'' + \ldots = 0,$$

where U is a convenient abbreviation for $Au_1 + Bu_2 + \ldots$. The result shows that the last expression is a solution. It should be noted that the proof is equally valid even when the coefficients are not constants. It is part of our policy to assume without proof that the number of independent constants A, B, &c., equals the order of the equation, though it is not difficult to prove by elimination that the number of linearly independent solutions cannot exceed the order. A proof for the second order will be found in an appendix to this chapter.

The discussion of the auxiliary equation is much facilitated by the use of the symbol D for d/dx, whence D^2 is d^2/dx^2, &c.; it largely obeys the ordinary laws of algebra. We have $D(u + v) = Du + Dv$, so that D obeys the distributive law. Also, if a is a constant,

$$(D + a)u = Du + au = au + Du = (a + D)u$$

and $Dau = aDu$, so that D is commutative with a constant. Moreover, $D^m D^n u = D^{m+n} u$, and the index law holds. Consider now $(D + \alpha)(D + \beta)u$, where α and β are constants.

Put
$$(D + \beta)u = v = Du + \beta u.$$

Then
$$\begin{aligned}(D + \alpha)(D + \beta)u &= (D + \alpha)v = Dv + \alpha v \\ &= D(Du + \beta u) + \alpha(Du + \beta u) \\ &= D^2 u + (\alpha + \beta)Du + \alpha\beta u \\ &= \{D^2 + (\alpha + \beta)D + \alpha\beta\}u.\end{aligned}$$

We are hence largely justified in using D symbolically and subjecting it to algebraical usage.

3, 2. *The Second-order Equation.*

The second-order equation is of such paramount importance that it warrants separate treatment. Its auxiliary equation can be written

$$ay + b\frac{dy}{dx} + c\frac{d^2y}{dx^2} = 0. \qquad \ldots\ldots \quad \text{(i)}$$

A curious and illuminating approach is afforded by considering the equation kinematically. Taking the displacement s as dependent on the time t, the equivalent kinematical form is

$$as + b\frac{ds}{dt} + c\frac{d^2s}{dt^2} = 0.$$

As ds/dt is the velocity v, the acceleration is

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}.$$

The equation thus becomes

$$as + bv + cv\frac{dv}{ds} = 0,$$

which is of the first order and homogeneous. In accordance with a remark in 2, 9·1, $v = ms$ is necessarily a solution provided m is a root of the quadratic

$$a + bm + cm^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Denoting the roots by α, β, we have the two intermediate solutions $v = \alpha s$, $v = \beta s$. On replacing v by ds/dt we get $s = e^{\alpha t}$, $s = e^{\beta t}$. These solutions being additive, we get the full solution as

$$s = Ae^{\alpha t} + Be^{\beta t},$$

where A, B are arbitrary constants.

Returning to orthodoxy, the equation (i) is normally written

$$(a + bD + cD^2)y = 0,$$

or the equivalent form

$$(D - \alpha)(D - \beta)y = 0,$$

where α, β are the roots of the quadratic (ii). We tentatively put $(D - \beta)y = z$, so that $(D - \alpha)z = 0$. Here the variables are separable and we have $z = c_1e^{\alpha x}$, so that $(D - \beta)y = c_1e^{\alpha x}$.

This last form is linear, and the integrating factor is $J = e^{-\beta x}$. Hence

$$ye^{-\beta x} = c_1\int e^{(\alpha-\beta)x}\,dx + B = \frac{c_1}{\alpha - \beta}e^{(\alpha-\beta)x} + B.$$

This gives
$$y = Ae^{\alpha x} + Be^{\beta x},$$

on writing A for $c_1/(\alpha - \beta)$.

A modification occurs when the quadratic (ii) for m has equal roots, so that $\alpha = \beta$. After determining J, we now have

$$ye^{-\beta x} = c_1\int dx + B = c_1x + B,$$

so that
$$y = (Ax + B)e^{\beta x},$$

on writing A for c_1. Certain other modifications are common in practice. If $b = 0$ and $a/c = -\alpha^2$, the quadratic in m has the equal but opposite real roots $\pm \alpha$. The solution is then

$$\begin{aligned} y &= Ae^{\alpha x} + Be^{-\alpha x} \\ &= A(\cosh \alpha x + \sinh \alpha x) + B(\cosh \alpha x - \sinh \alpha x) \\ &= P \cosh \alpha x + Q \sinh \alpha x, \end{aligned}$$

on writing P, Q for the arbitrary $A \pm B$. Alternatively, if b is zero and $a/c = +\alpha^2$ the quadratic (ii) has the equal and opposite imaginary roots $\pm i\alpha$. The solution is then $y = Ae^{i\alpha x} + Be^{-i\alpha x}$, which, by the use of De Moivre's theorem, can equally well be written

$$y = P \cos \alpha x + Q \sin \alpha x.$$

Here P has replaced $A + B$ and Q has replaced $i(A - B)$. In spite of the presence of i this last term is not necessarily imaginary since A, B are arbitrary. In electrical work this last form is frequently written with an amplitude and a phase angle as $y = R \sin (\alpha x + \phi)$. The two arbitrary constants are now R, ϕ, and the connexion is

$$\begin{aligned} P &= R \sin \phi, \qquad & R &= \sqrt{(P^2 + Q^2)}, \\ Q &= R \cos \phi, \qquad & \tan \phi &= P/Q. \end{aligned}$$

When the quadratic gives the complex roots $\alpha \pm i\beta$, the formal solution is

$$\begin{aligned} y &= A \exp (\alpha + i\beta)x + B \exp (\alpha - i\beta)x \\ &= e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}) \\ &= e^{\alpha x}(P \cos \beta x + Q \sin \beta x) \\ &= Re^{\alpha x} \sin (\beta x + \phi). \end{aligned}$$

In this case α is almost invariably negative in practice. The amplitude R is invariably taken as positive. The phase ϕ is usually taken to be less than π, and may be positive or negative, according to circumstances.

3, 2·1. *Avoidance of Imaginaries.*

Any misgivings that one may feel about the use of imaginaries in the above discussion can be lulled as follows. The case of $m = \pm i\alpha$ corresponds to the equation

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0.$$

By the insertion of the factor $2dy/dx$, we have

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2a^2y\frac{dy}{dx} = 0.$$

Integration gives $$\left(\frac{dy}{dx}\right)^2 + a^2y^2 = a^2R^2,$$

using a^2R^2 as the arbitrary constant. The variables are now separable,

$$\frac{dy}{\sqrt{(R^2 - y^2)}} = a\,dx,$$

$$\sin^{-1}\frac{y}{R} = ax + \phi, \quad \text{and} \quad y = R\sin(ax + \phi)$$

as before. Similarly, the case of $m = \alpha \pm i\beta$ arises from the equation

$$\frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + (\alpha^2 + \beta^2)y = 0.$$

The substitution $y = e^{\alpha x}z$ implies

$$Dy = e^{\alpha x}(D + \alpha)z; \quad D^2y = e^{\alpha x}(D + \alpha)^2z.$$

We thus have

$$e^{\alpha x}(D + \alpha)^2z - 2\alpha e^{\alpha x}(D + \alpha)z + (\alpha^2 + \beta^2)e^{\alpha x}z = 0,$$

or $$(D^2 + \beta^2)z = 0,$$

which we know leads without the use of imaginaries to

$$z = R\sin(\beta x + \phi).$$

Hence the solution is as before

$$y = Re^{\alpha x}\sin(\beta x + \phi).$$

3, 3. Evidently in practice the quadratic 3, 2 (ii) in m is superfluous. Having agreed to treat D symbolically, we need have no compunction in using $a + bD + cD^2 = 0$, with the roots $D = \alpha, \beta$. As an example of the method, consider the problem of finding a solution of the equation

$$\frac{d^2y}{dx^2} + 1{\cdot}62\frac{dy}{dx} + 2{\cdot}34y = 0$$

which shall pass through the origin and make an angle of 68° with OX. Since the square of $\frac{1}{2}(1{\cdot}62)$ is definitely less than 2·34, the roots of

$$D^2 + 1{\cdot}62D + 2{\cdot}34 = 0$$

are complex, and we have

$$D = -0{\cdot}81 \pm i1{\cdot}30$$

to two places. The general solution is

$$y = e^{-\alpha x}(P\cos\beta x + Q\sin\beta x).$$

with $\alpha = 0{\cdot}81$, $\beta = 1{\cdot}30$. The curve is to pass through the origin, and the simultaneous values $x = 0 = y$ show that P is zero. The modified form

$$y = Qe^{-\alpha x}\sin\beta x$$

gives
$$\frac{dy}{dx} = Qe^{-\alpha x}(\beta\cos\beta x - \alpha\sin\beta x).$$

At the origin we are to have $dy/dx = p = \tan 68°$. We deduce that $p = Q\beta$, whence $Q = \beta^{-1}\tan 68° = 2{\cdot}48/1{\cdot}30 = 1{\cdot}91$. The required solution is therefore

$$y = 1{\cdot}91\exp(-0{\cdot}81x)\sin 1{\cdot}30x.$$

EXERCISES

1. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 6y.$ $[y = Ae^{2x} + Be^{-3x}.]$

2. $y + 4\dfrac{dy}{dx} + 4\dfrac{d^2y}{dx^2} = 0.$ $[y = (A + Bx)e^{-\frac{1}{2}x}.]$

3. $(D^2 + 4D + 13)y = 0.$ $[y = e^{-2x}R\sin(3x + \varphi).]$

4. Find a solution of $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 2y = 0$ which shall pass through the origin. $[y = A(e^{-2x} - e^{-\frac{1}{2}x}).]$

5. Prove that no solution of $(a + bD + cD^2)y = 0$ can touch OX at the origin.

3.4. *Applications.*

We can now exemplify the foregoing principles by some applications.

1. *Simple Harmonic Motion.*

Probably the simplest is the kinematical problem of a moving point P describing with constant angular velocity ω counterclockwise a circle of centre O and radius R in a vertical plane. If the moving radius OP has described an angle θ, as measured from the downward vertical through O, we have $\dot{\theta} = \omega$, and the displacement, in plan, of P from the middle is $x = R \sin \theta$. This is to the right or left according as OP is right or left of the vertical. The point P has tangential velocity $R\omega$ but no tangential acceleration. Similarly, it has no normal velocity; but the normal acceleration is $R\omega^2$ radially inward. Hence in plan the velocity $\dot{x}$ is $R\omega \cos \theta$, which is directed left or right according as OP is above or below the horizontal; and the acceleration $\ddot{x}$ is $- R\omega^2 \sin \theta$, which is directed left or right according as OP is right or left of the vertical, so that it is always directed to the middle.

The result shows that $\ddot{x} + \omega^2 x = 0$.

The plan of P describes a simple harmonic motion of amplitude R and frequency $\omega/2\pi$, and this is its differential equation. If we suppose the stop-watch started when OP had already described an angle ϕ, then at any subsequent time t we have $\theta = \phi + \omega t$. It will be observed that the differential equation takes no cognizance of either R or ϕ, so that it would equally apply to any point on OP, or to any point on any radius at a fixed inclination to OP. The solution from the above is $x = R \sin \theta$, or $x = R \sin (\omega t + \phi)$, where R, ϕ must remain arbitrary unless conditions are specified for their determination. This accords with our previous work, for the equation can be written $(D^2 + \omega^2)x = 0$, which has the equal and opposite imaginary roots $D = \pm i\omega$.

2. *Conduction of Heat in a Bar.*

As a second example, consider a uniform unlagged bar, of which one end is maintained at a constant temperature. We assume that at distance λ from the hot end the bar is at room temperature and that matters have reached a steady state. This last expression means that the temperature at any point has ceased to vary. Let $H(A)$ denote the heat per second that crosses some section A at distance x from the hot end. Similarly, let $H(B)$ apply to a section B at distance $x + dx$. Then $H(B)$ is certainly not greater than $H(A)$; and if it is less, the difference must be accounted for by emission from the part AB. Let θ be the temperature at A, then $H(A)$ is proportional to the negative temperature gradient $-d\theta/dx$. The negative sign is necessary since the heat must flow from high temperature to low. The other factors affecting $H(A)$ are such constants as the area of the cross-section, the specific heat and the conductivity. We can accordingly write $H(A) = -c_1 \, d\theta/dx$. The temperature at B will be $\theta + d\theta = \theta + (d\theta/dx)dx$. This does not imply that it is greater at B than at A;

it is merely a statement that it may possibly be different since the distance is different. We thus have

$$H(B) = -c_1 \frac{d}{dx}\left[\theta + \frac{d\theta}{dx}\right] dx.$$

The heat emitted by AB is accordingly

$$H(A) - H(B) = +c_1 \frac{d^2\theta}{dx^2} dx.$$

So far nothing has been said about the scale of temperature, nor does it matter. If we take the temperature θ at A as being measured above room-temperature, then the emission of heat per second by AB is proportional to θ. The other factors affecting the amount are the surface area (perimeter $\times$ dx) and an emissivity constant. The net result is $c_2\theta\, dx$ where c_2 is some constant. Equating the two forms of the heat emitted, we have

$$c_1 \frac{d^2\theta}{dx^2} dx = c_2\theta\, dx, \quad \text{or} \quad \frac{d^2\theta}{dx^2} = \alpha^2\theta,$$

whence $(D^2 - \alpha^2)\theta = 0$. This has the roots $D = \pm\alpha$, and the solution is

$$\theta = P \cosh \alpha x + Q \sinh \alpha x.$$

We can determine the constants, for if T is the excess temperature at the hot end, we have $\theta = T$ when $x = 0$, so that

$$T = P \cosh 0 + Q \sinh 0 = P.$$

At the cool end we have $x = \lambda$, $\theta = 0$, so that

$$0 = P \cosh \alpha\lambda + Q \sinh \alpha\lambda.$$

Thus $P = T$ and $Q = -T \coth \alpha\lambda$, leading to

$$\theta = T \cosh \alpha x - T \coth \alpha\lambda \sinh \alpha x = \frac{T \sinh \alpha(\lambda - x)}{\sinh \alpha\lambda}.$$

The temperature at any point is proportional to $\sinh \alpha y$ where y is measured from the cool end. It will appear later that a similar analysis applies to a leaky transmission line when discussing the voltage with one end earthed; or the current with one end insulated.

3. *Damped Harmonic Motion.*

Spring.—Turning to the more complicated cases we consider a spring of stiffness k, lying on a rough horizontal table, with the left end fixed. The right end is attached to a mass m which is moving to the right when x is the extension of the spring. The velocity of m is $\dot{x}$ and the acceleration $\ddot{x}$, both measured positive to the right. The stiffness being defined as the force to give unit extension, the spring exerts a force kx on the mass, directed to the left. We allow for the roughness of the table by taking a frictional resistance proportional to the velocity, or $p\dot{x}$ directed to the left. The equation of motion is then

$$m\ddot{x} = -p\dot{x} - kx,$$

or

$$m\ddot{x} + p\dot{x} + kx = 0. \qquad \text{(i)}$$

If p is zero we have simple harmonic motion; otherwise the motion is said to be damped.

Electrical Analogue.—The equation (i) has important applications in electricity. Before discussing its solution, we consider the following problem.

A condenser of capacity C farads discharges through a series circuit of resistance R ohms and inductance L henries. If V be the voltage of the condenser at any time t, the amount of electricity in the condenser is Q coulombs, $=CV$. The current I amperes flowing out of the condenser is the rate of diminution of Q, so that

$$I = -\frac{dQ}{dt} = -C\frac{dV}{dt}.$$

From this we derive the further relation $dI/dt = -Cd^2V/dt^2$. Ohm's law gives

$$RI = V - L\frac{dI}{dt},$$

and if we multiply by C and differentiate, we have

$$CR\frac{dI}{dt} = -I - CL\frac{d^2I}{dt^2},$$

or

$$CL\frac{d^2I}{dt^2} + CR\frac{dI}{dt} + I = 0. \qquad \text{(ii)}$$

The elimination of I from Ohm's law gives

$$-CR\frac{dV}{dt} = V + CL\frac{d^2V}{dt^2},$$

or

$$CL\frac{d^2V}{dt^2} + CR\frac{dV}{dt} + V = 0. \qquad \text{(iii)}$$

It thus appears that I and V satisfy the same equation, and with a change of notation all three equations (i), (ii) and (iii) have the form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + n^2x = 0,$$

with $$2b = \frac{R}{L} \text{ or } \frac{p}{m}, \quad n^2 = \frac{1}{CL} \text{ or } \frac{k}{m};$$

or $$(D^2 + 2bD + n^2)x = 0,$$

with the roots $D = -b \pm \sqrt{(b^2 - n^2)}$. This gives three cases to be considered according as b is greater than, equal to, or less than n.

Case (i): $b > n$. We put $b^2 - n^2 = c^2$. This implies $b > c$, and the two roots $D = -b \pm c$ are both negative. The solution has the forms

$$\begin{aligned} x &= A\exp(-b + c)t + B\exp(-b - c)t \\ &= e^{-bt}(Ae^{ct} + Be^{-ct}) \\ &= e^{-bt}(P\cosh ct + Q\sinh ct). \end{aligned}$$

There is evidently no oscillation, and the first form shows that as t becomes large, x approaches zero since both the negative exponentials tend to zero. The third form shows that x cannot be zero more than once and may never be zero at all. For $x = 0$ implies $\tanh ct = -P/Q$, and there is one solution or none according as the absolute value of P/Q is less than or greater than unity. In practice this corresponds pretty well with the motion of the light-spot when using a dead-beat galvanometer. The constants A, B or P, Q can be determined when two conditions are known, e.g. the initial displacement and the velocity.

Case (ii): $b = n$. The two roots are now $D = -b, -b$, and the solution is $x = (A + Bt)e^{-bt}$. As t becomes indefinitely large, e^{-bt} tends to zero, and as te^{-t} can be written

$$\frac{t}{e^t} = \frac{1}{t^{-1} + 1 + \frac{1}{2}t + \ldots}$$

it also tends to zero. Here again there cannot be more than one value of t for which x is zero, and there may be none within the period of observation. The damping is known as critical, in the sense that any reduction in its value permits oscillation, as will appear in the next few lines.

Case (iii): $b < n$. This is easily the most important of the three. As $b^2 - n^2$ is now negative the roots are complex. If we put $n^2 - b^2 = c^2$, so that n is greater than c, the roots are $D = -b \pm ic$, and the solution is

$$x = e^{-bt} R \sin(ct + \phi). \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

The argument $(ct + \phi)$ steadily increases with time, so that the trigonometrical term is alternately positive and negative and the motion is oscillatory; but the amplitude $e^{-bt}R$ steadily decreases with time, and the oscillation (like all natural vibrations) is gradually damped out of existence. The time interval between two successive transits in the same direction through the zero-mark, which implies $\sin(ct + \phi) = 0$, is given by a change of 2π in the argument $(ct + \phi)$, i.e. a change of $2\pi/c$ in t. This is the periodic time and is unrelated to the amplitude. Had the damping been zero, the motion would have been simple harmonic, with period $2\pi/n$. As mentioned above, n is greater than c, so that one effect of the damping is to increase the period, or slow down the frequency.

The equation is a very fair approximation to the motion of the light-spot when a galvanometer needle is being resisted by the atmosphere. The positions of maximum displacement, left or right, are instants of temporary rest. If we calculate the velocity, we have

$$\begin{aligned} \dot{x} &= Re^{-bt}[c\cos(ct + \phi) - b\sin(ct + \phi)] \\ &= -Rne^{-bt}\sin(ct + \phi - \gamma), \quad n^2 = b^2 + c^2, \quad \tan\gamma = \frac{c}{b}. \end{aligned}$$

The instants of temporary rest are given by $\sin(ct + \phi - \gamma) = 0$, so that the time interval between two consecutive instants of rest on the same side of the zero corresponds to a change of 2π in the argument $(ct + \phi - \gamma)$, i.e. a change of $2\pi/c$ as before in t. It is therefore immaterial whether the period be measured between consecutive instants of rest on the same side of the zero, or between consecutive transits through the zero in the same direction.

This leads to an important constant connected with the motion. Let x_1, x_2 be two consecutive maximum displacements on the same side. The time-interval between them is a period $\tau = 2\pi/c$. Hence if

$$x_1 = R\exp(-bt)\sin(ct + \phi),$$

then

$$x_2 = R\exp\{-b(t + \tau)\}\sin(ct + \phi + 2\pi).$$

This gives $\quad x_1/x_2 = \exp b\tau, \quad$ or $\quad \log(x_1/x_2) = b\tau = 2\pi b/c.$

This important factor $2\pi b/c$ is known as the "*logarithmic decrement*". It figures prominently in telephony and acoustics. If the time-interval for a given number of swings be taken, together with a note of the first and last amplitudes, the period and the damping can be computed.

In considering the x, t graph of (iv) with t horizontal and x vertical, the curve somewhat resembles a sine curve; but there is a good deal of distortion, though there is no loss of periodicity. Whereas a pure sine curve $x = R \sin(ct + \phi)$ would lie between the parallel lines $x = \pm R$, the present curve lies between the negative exponential curves $x = \pm Re^{-bt}$. The initial displacement is the intercept on the vertical axis, given by $t = 0$, so that its value is $R \sin \phi$. This is positive or negative according as ϕ is positive or negative; and the curve slopes initially upwards or downwards according as the initial velocity is positive or negative. The intercept increases as ϕ increases from zero to $\frac{1}{2}\pi$, thus throwing the curve to the left, so that of two similar and adjacent curves the one on the left is really leading, in the electrical sense. The intercept on the horizontal axis, corresponding to the instant of passing the zero-mark, is given by $\sin(ct + \phi) = 0$, so that $t = -\phi/c$. It is instructive to sketch the curve roughly for various types of initial conditions.

The following is an example of numerical work. A series circuit has a capacity of 2 μF, an inductance 0·86 mH and a variable resistance. Calculate the resistance when the circuit just fails to be oscillatory, and also when it is oscillatory at 3500 cycles per second. The voltage and current both depend on the equation

$$CLD^2 + CRD + 1 = 0.$$

If the damping is critical, this has equal roots, for which the condition reduces to $R^2C = 4L$. As $C = 2\ \mu\text{F} = 2 \cdot 10^{-6}$ F and $L = 0{\cdot}86$ mH $= 8{\cdot}6 \cdot 10^{-4}$ H, we have $R = 41{\cdot}4$ ohms as the critical resistance. Otherwise we write

$$D^2 + \frac{R}{L} D + \frac{1}{CL} = 0,$$

and complete the square as

$$\left(D + \frac{R}{2L}\right)^2 = \frac{R^2}{4L^2} - \frac{1}{CL}.$$

If the circuit is oscillatory, the roots are $D = -b \pm ic$, where

$$b = \frac{R}{2L}, \quad c^2 = \frac{1}{CL} - \left(\frac{R}{2L}\right)^2,$$

and the frequency of oscillation is $c/2\pi$ cycles per second, given as $3500 \sim$. Hence $c^2 = (2\pi \,.\, 3500)^2 = 4{\cdot}836 \,.\, 10^8$. Also $1/CL = 5{\cdot}813 \,.\, 10^8$, so that $(R/2L)^2 = 97{\cdot}7 \,.\, 10^6$ and $R/2L = 9885$, leading to $R = 17$ ohms.

EXERCISES, 3, 5

1. Verify that the solution of

$$1{\cdot}7\ddot{x} + 8{\cdot}3\dot{x} + 5{\cdot}5x = 0$$

is non-oscillatory. If the initial conditions are $t = 0$, $x = 7{\cdot}3$, $\dot{x} = -3{\cdot}2$, prove that x has neither a maximum nor a finite zero, but approaches zero steadily.
[$x = 8{\cdot}08 \exp(-0{\cdot}79t) - 0{\cdot}78 \exp(-4{\cdot}09t)$.]

2. In a series circuit the current is oscillatory with logarithmic decrement n. Prove that

$$\frac{L}{CR^2} = \left(\frac{\pi}{n}\right)^2 + \frac{1}{4}.$$

Explain this result when the resistance approaches zero. Numerically $R = 5$ ohms, $L = 0{\cdot}3$ mH, $n = 0{\cdot}4$. Calculate C.

In general, if $D = -b \pm ic$, and the condenser has initial voltage V_0, prove that its voltage V at any subsequent time t is given by

$$V \sin\varphi = V_0 e^{-bt} \sin(ct + \varphi),$$

where $\tan\varphi = c/b$. [$C = 0{\cdot}19$ μF.]

3. A mass of 10 lb. moves on a rough horizontal table attached to a spring of stiffness 0·42 lb./in. The resistance is 2·3 lb. per ft./sec. velocity. Initially, $t = 0$, $x = -5$ in., $\dot{x} = 2{\cdot}7$ ft./sec. Deduce that the periodic time is 3·91 sec., $R = 9{\cdot}9$ in., the phase angle φ is in the fourth quadrant and equals $-30°$. Note that the force must be in absolute units, and if g be taken as 32, the length unit is the foot.

4. Assuming that in the motion $\ddot{x} + 2b\dot{x} + n^2x = 0$, the damping is above the critical value, let the initial conditions be $t = 0$, $x = a$, $\dot{x} = u$. Prove that x necessarily has a maximum value. If u be replaced by $-u$, show that x does not pass through the zero if the absolute value of u lies between $a(b \pm c)$, where $c^2 = b^2 - n^2$; but that x passes through zero if u is negative and numerically greater than $a(b + c)$.

5. If in the last example the damping is critical and the initial conditions are $t = 0$, $x = a$, $\dot{x} = -u$, find the greatest value of u if x does not pass through the zero.

6. Prove that the contacts of the curve $x = Re^{-bt}\sin(ct + \varphi)$ with its boundary curve $x = Re^{-bt}$ occur at intervals of a period and slightly after the instants of temporary rest.

If the light-spot swings from zero out to the right and back to zero, prove that the first half-swing is quicker than the return half.

7. The motion of a 10-lb. mass is controlled by a spring of stiffness 8 lb./in. A dashpot mechanism provides a damping of p lb. per ft./sec. Estimate p if a single swing reduces the amplitude to a fifth of its value. [Ans. [illegible]5 approx.]

8. A uniform unlagged bar of length λ has the end $x = 0$ maintained at temperature T_1, and the end $x = \lambda$ maintained at temperature T_2. Prove that after a considerable time the temperature distribution is given by

$$\frac{T_1 \sinh \alpha(\lambda - x) + T_2 \sinh \alpha x}{\sinh \alpha\lambda}.$$

3, 6. *The Complementary Function in General.*

When the degree of the auxiliary equation exceeds two, we have

$$(a + bD + cD^2 + \ldots)y = 0,$$

which we may denote by $F(D)y = 0$. The procedure is then mainly a repetition of previous work. If $F(D)$ has the factor $(D - \alpha)$ corresponding to the real root α of the equation $F(D) = 0$, we can put $F(D) = \{\phi(D)\}(D - \alpha)$. Any value of y that makes $(D - \alpha)y$ zero must give

$$F(D)y = \{\phi(D)\}(D - \alpha)y = \phi(D)0 = 0.$$

This means that $y = Ae^{\alpha x}$ is part of the complementary function, and as solutions are additive the full value of the complementary function is

$$Ae^{\alpha x} + Be^{\beta x} + Ce^{\gamma x} + \ldots,$$

where A, B, C, &c., are arbitrary, and α, β, γ, &c., are the roots of $F(D) = 0$.

If a root is repeated the corresponding factor is $(D - \alpha)^2$, and we can put

$$F(D) = \{\psi(D)\}(D - \alpha)^2.$$

The value of y that makes $(D - \alpha)^2 y = 0$ must make $F(D)y = 0$, i.e. $(A + Bx)e^{\alpha x}$ is part of the complementary function. By an obvious extension, a factor $(D - \alpha)^3$ would give the solution $(A + Bx + Cx^2)e^{\alpha x}$.

Imaginary and complex roots present no new difficulties. The two roots $\pm i\alpha$ correspond to a factor $(D^2 + \alpha^2)$, and by writing

$$F(D)y = \{\phi(D)\}(D^2 + \alpha^2)y,$$

we conclude that $(P \cos \alpha x + Q \sin \alpha x)$ is part of the solution. Similarly, the complex roots $\alpha \pm i\beta$ correspond to a factor $\{D^2 - 2\alpha D + (\alpha^2 + \beta^2)\}$, and by writing

$$F(D)y = \{\psi(D)\}\{D^2 - 2\alpha D + (\alpha^2 + \beta^2)\}y,$$

we conclude that $e^{\alpha x}(P\cos\beta x + Q\sin\beta x)$ is part of the solution. The rule then evidently is to factorize $F(D)$ into real linear or quadratic factors, and write down the solution from these. In practice this may involve a good deal more labour than text-book examples would imply. It would be no five-minute job to solve, say, $y''' + y' + 7y = 0$.

Example.—$\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0.$

We have $(D^3 - D^2 + D - 1)y = 0 = (D - 1)(D^2 + 1)y.$

The roots are $1, \pm i$, and the solution is therefore

$$y = Ae^x + B\sin x + C\cos x.$$

EXERCISES

1. $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} + 2y = 0.$ $[y = Ae^{-2x} + (B + Cx)e^x.]$

2. $y''' - y'' - 4y' + 4y = 0.$ $[y = Ae^x + P\cosh 2x + Q\sinh 2x.]$

3. Prove that any solution of the equation $y''' + y'' + 4y' + 4y = 0$, which contains three non-zero constants, must cross OX an infinite number of times at intervals which gradually tend to equality with increasing x. $[y = Ae^{-x} + R\sin(2x + \varphi).]$

4. Find a solution of $y''' - y'' - y' + y = 0$ which passes through the origin.

Prove that any solution of this equation which touches OX at the origin can never meet it again. $[y = Axe^x + B\sinh x.]$

5. Accepting the statement that $(A + Bx)e^{\alpha x}$ is part of the complementary function if there is a factor $(D - \alpha)^2$, verify the statement in the text, that a factor $(D - \alpha)^3$ leads to $(A + Bx + Cx^2)e^{\alpha x}$.

6. Prove that if $F(D) = D^2\varphi(D)$, then $A + Bx$ is part of the complementary function. Generalize the result.

7. Solve the equation $(D^4 - \omega^4)y = 0$ which occurs in the theory of shafts. If the constants are determined by the conditions that $y = 0 = y'$ both when $x = 0$ and when $x = \lambda$, prove that ω and λ are related by the equation $\cos\omega\lambda = \operatorname{sech}\omega\lambda$. Deduce from a rough sketch that this transcendental equation has the approximate roots $\omega\lambda = 3\pi/2, 5\pi/2, \ldots$.
$[y = A\cos\omega x + B\sin\omega x + C\sinh\omega x + D\cosh\omega x.]$

8. Find the complementary function of the equation

$$(D^4 + \omega^4)y = cx,$$

which has several practical applications.

9. $(D^4 + 2n^2D^2 + n^4)y = 0.$

10. $(D^4 - 3D^3 + 3D^2)y = 0.$

11. $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0.$

12. $\dfrac{d^2y}{dx^2} - 4pq\dfrac{dy}{dx} + (p^2 + q^2)^2y = 0.$

3, 7. *The Particular Integral.*

It has already been remarked that the auxiliary equation $F(D) = 0$, which determines the complementary function, furnishes all the arbitrary constants that the equation 3, 1 (i) demands. The particular integral can be any solution whatever, the simpler the better.

Before proceeding to the methods of determination for the commoner cases we make a few observations of general applicability. It may happen that $f(x)$ is the sum of a number of functions, so that the equation 3, 1 (i) has the form

$$F(D)y = X_1 + X_2 + X_3 + \dots .$$

In this case it is legitimate to treat the functions separately; for if y_1 is a solution when we use only X_1, we have $F(D)y_1 = X_1$. Similarly with a change of suffix we have $F(D)y_2 = X_2$, and so on. Summation gives

$$F(D)(y_1 + y_2 + \dots) = X_1 + X_2 + X_3 + \dots,$$

so that the sum of the partial solutions $y_1 + y_2 + \dots$ is a solution. It may happen in this process that terms appear which are already in the complementary function. In that case they can be rejected; for since we have an arbitrary number of them in the complementary function, there is no point in keeping more of them in the particular integral. Conversely we may, when the occasion suits, add to the particular integral terms occurring in arbitrary amounts in the complementary function.

3, 8. *$f(x)$ a Polynomial.*

Turning now to the particular methods, we begin with the case where $f(x)$ is a polynomial, so that the equation has the form

$$(a + bD + cD^2 + \dots)y = L + Mx + Nx^2 + \dots . \quad \text{(i)}$$

Consider the result of letting $F(D)$ operate on an arbitrary polynomial, i.e.

$$(a + bD + cD^2 + \dots)(p + qx + rx^2 + \dots).$$

The first operator a produces a polynomial of the same degree. The second operator bD produces a polynomial of degree one lower. The third operator produces a polynomial of degree two lower than the original, and so on. If the process goes on long enough we merely produce a constant, and thereafter achieve complete annihilation.

The net result is evidently a polynomial of the same degree as the original, and by a proper choice of the coefficients p, q, r, &c., the result can be made to coincide with the $f(x)$ of the given equation. We are accordingly justified in assuming that y is a polynomial of the same degree as $f(x)$.

Example 1. $(3 + 2D - D^2 - D^3)y = 9 - 2x + 3x^2$. Here $f(x)$ is quadratic, and we assume y to be the quadratic $p + qx + rx^2$. Substitution gives

$$3(p + qx + rx^2) + 2(q + 2rx) - 2r - 0 = 9 - 2x + 3x^2.$$

A comparison of the coefficients of the various powers of x leads to the simple simultaneous equations

$$3r = 3,$$
$$3q + 4r = -2,$$
$$3p + 2q - 2r = 9.$$

We have in succession $r = 1$, $q = -2$, $p = 5$, and the particular integral is $y = 5 - 2x + x^2$.

The reader can satisfy himself on two points that emerge from this. In the first case it would have been pointless to assume that y was cubic rather than quadratic; and in the second case, having assumed that y was quadratic, the operators above D^2 can be ignored. The sequel is that if $f(x)$ is a polynomial of degree zero, reducing to the mere constant L, then y itself is a constant, in which case all powers of D can be ignored. The result of the substitution in (i) would be $ap = L$, whence $p = L/a$. Another and minor point is that the coefficient of the highest power in y is obtained by dividing the highest power in $f(x)$ by a.

Example 2.—A vertical spring of stiffness k has its upper end fixed. The lower end carries a mass m. If at any time t the extension of the spring is x, the equation of motion is $m\ddot{x} = mg - kx$, or $(mD^2 + k)x = mg$. The complementary function gives the simple harmonic motion $x = R\sin(\omega t + \varphi)$, where $\omega^2 = k/m$. The particular integral gives $x = mg/k$, the physical interpretation being that it is the extension in the equilibrium position.

An anomaly arises when some of the leading terms in $F(D)$ are absent, e.g. $a + bD$, so that the equation (i) reads

$$(cD^2 + dD^3 + \ldots)y = L + Mx + \ldots.$$

Here two direct integrations are possible, and if the above argument were applied directly it would no longer be true that the assumption of a polynomial for y would produce a polynomial of the same degree. The anomaly is more apparent than real, for y can still be taken as a polynomial, though the degree will necessarily have to be higher. But it is useless now to include the terms $p + qx$ as they would be annihilated by the operator $F(D)$. Their omission is compensated by their appearance as part of the complementary function arising from the repeated roots $D = 0, 0$. We accordingly assume $y = rx^2 + sx^3 + \ldots$. The reader is left to formulate the rule as to the highest power employed.

Example 3.—$\dfrac{d^4y}{dx^4} - 3\dfrac{d^3y}{dx^3} = 9x + 15 = D^3(D-3)y.$

The complementary function is $A + Bx + Cx^2 + He^{3x}$. In finding the particular integral we ignore the quadratic portion and assume $y = sx^3 + tx^4$. Substitution then gives

$$24t - 3(6s + 24tx) = 9x + 15,$$

whence $s = -1$, $t = -1/8$. The full solution is thus

$$y = A + Bx + Cx^2 - x^3 - \frac{x^4}{8} + He^{3x}.$$

EXERCISES

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 8(x-2).$ [$y = Ae^x + Be^{2x} + 4x - 2.$]

2. Prove that if the equation has the form

$$(aD^n + bD^{n+1} + \ldots)y = \alpha + \beta x + \ldots + \mu x^m,$$

the correct assumption is

$$y = Ax^n + Bx^{n+1} + \ldots + Hx^{n+m}.$$

Hence determine the complete solution of

$$3\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 12x^2 + 2.$$

$$\left[y = A + Bx + 14x^2 - 3x^3 + \tfrac{1}{2}x^4 + C\exp\left(-\frac{2x}{3}\right).\right]$$

3. Assuming the equation $L\dfrac{dI}{dt} + RI = E$ proved in 2, 3, Ex. 3, find the complementary function and the particular integral. Deduce the result

$$I = \frac{E}{R}\left\{1 - \exp\left(-\frac{Rt}{L}\right)\right\}$$

for the current I at t sec. after closing the circuit. Treat similarly the problem of charging a condenser; see 2, 5, Ex. 13.

4. Prove that if the equation has the form

$$(a + bD^3 + cD^4 + \ldots)y = \alpha + \beta x + \gamma x^2,$$

the particular integral is $y = (\alpha + \beta x + \gamma x^2)/a.$

5. One of the curves

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 2x + 4$$

touches OX at the origin. What is its equation?

[$18y = 20e^{-3x} - 27e^{-2x} + 6x + 7$.]

6. A curve of the family $y'' + 5y' + 5y = 6$ crosses OY with an intercept 0·75 and a down slope $-\tan^{-1} 1{\cdot}5$. Prove that it has a horizontal asymptote which it crosses only once, and that it has a minimum value and an inflection. Does it cross OX? [$y = 0{\cdot}95 \exp(-3{\cdot}618x) - 1{\cdot}40 \exp(-1{\cdot}382x) + 1{\cdot}2$.]

7. Prove that the equation

$$(a + bD + cD^2 + \ldots)y = L$$

can be solved by taking a new variable $u = y - \dfrac{L}{a}$.

8. In studying the strains set up in large storage tanks we meet the equation $D^4y + \omega^4y = cx$. Integrate it. See 3, 6, No. 8.

9. Find the conditions that the particular integral of

$$(a + bD + cD^2)y = L + Mx + Nx^2$$

shall consist of a single term. [$a(L + Mx) = 2N(bx + c)$.]

10. Find the particular integral of

$$\frac{d^2x}{dt^2} + 11\frac{dx}{dt} + 28x = 84t^2 - 242t - 591.$$

[$x = 3t^2 - 11t - 17$.]

3, 9. *Practical Applications.*

As an example of the practical application we take the theory of beams. Consider a horizontal cantilever of length λ with the left end encastrée (fixed, built in), the right end bearing a load W. The principles of statics dictate that there must be a vertical reaction of magnitude W at the wall. These two constitute a couple $W\lambda$ clockwise. There is accordingly a counter-acting couple C at the wall counterclockwise, whose function is to restrain the left end of the beam from rising. If that were the whole story we should have $C = W\lambda$; but if we assume a horizontal thrust P at the right, there must be an opposing thrust P to the right at the wall. These two comprise a clockwise couple Pd, where d is the deflection at the end. We now have $C = W\lambda + Pd$, a single equation with the two unknowns C and d; the problem is statically indeterminate and recourse must be had to the theory of elasticity.

We take the origin at the wall and the horizontal undeflected centre line as x axis. The deflection y at any point is measured positive downwards, and as y increases with x, we have y' positive. Incidentally it is so small that nothing short of optical or electrical methods will measure it; its square is accordingly negligible. As y' also increases with x, we have y'' positive. The formula for the radius of curvature being $\rho = (1 + y'^2)^{3/2}/y''$, we have the curvature (the reciprocal of the radius of curvature) approximately given by y''. The Euler theory of beams takes the curvature at any point of the neutral axis as proprotional to the bending moment acting over the corresponding normal section of the beam. This bending moment is accordingly given with sufficient accuracy for ordinary purposes by the expression EIy''. Here E is Young's modulus for the material and I is the modulus of the cross-section, a sort of moment of inertia. The "proof" can be found in any text on the strength of materials.

We now take a normal section at any point A of the beam distant x to the right of the wall, and consider the influence that the right portion exercises on the left portion. The upper fibres are in tension, and therefore exert a pull to the right. Conversely, the lower fibres are in compression and exert a thrust to the left. The net effect is a clockwise couple. This is the couple measured by EIy''.

The horizontal equilibrium of the left portion further demands that there shall be a thrust P acting to the left at A to counterbalance the thrust P from the wall. These two give a clockwise couple Py, where y is the deflection of the neutral axis at A. Similarly, there must be a downward force W at A to give vertical equilibrium with the upthrust W at the wall. These two constitute a clockwise couple Wx. The equation of equilibrium of the left portion becomes

$$EIy'' = C - Py - Wx. \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

As a working rule it is as well in any particular case to begin by writing down $EIy'' = C$, even when C is non-existent. The remaining couples or moments are then placed on the right with sign to accord with C, which is positive counterclockwise. As for the solution, we have

$$EIy'' + Py = C - Wx = EI(D^2 + n^2)y,$$

where $n^2 = P/EI$, so that the complementary function is

$$A \cos nx + B \sin nx.$$

The particular integral must be linear and so D^2 can be ignored. Assuming $y = p + qx$, we have

$$Py = P(p + qx) = C - Wx,$$

whence

$$y = p + qx = \frac{C - Wx}{P}.$$

The full solution is therefore

$$y = A\cos nx + B\sin nx + \frac{C - Wx}{P}. \quad . \quad . \quad . \quad \text{(ii)}$$

The determination of the arbitrary A, B is carried out from known conditions. There is no deflection at the wall, and $x = 0$ implies $y = 0$;

hence $$0 = A + \frac{C}{P}, \quad A = -\frac{C}{P}.$$

Moreover, the beam projects horizontally, so that $x = 0$ implies $y' = 0$. Differentiation of (ii) gives

$$y' = nB\cos nx - nA\sin nx - \frac{W}{P},$$

so that the end condition gives

$$0 = nB - \frac{W}{P}, \quad B = \frac{W}{nP}.$$

The solution now is

$$y = \frac{C}{P}(1 - \cos nx) + \frac{W}{nP}(\sin nx - nx). \quad . \quad . \quad \text{(iii)}$$

We still have the two unknowns C and d; but we have a further condition that there is no couple applied to the end of the beam, so that $y'' = 0$ when $x = \lambda$. The substitution of these values in equation (i) gives $C = Pd + W\lambda$, a result already known from statics and serving as a check on our work. If we apply our solution (iii) to the end of the beam, we have

$$d = \frac{Pd + W\lambda}{P}(1 - \cos n\lambda) + \frac{W}{nP}(\sin n\lambda - n\lambda).$$

With a little algebraic reduction this leads to

$$\frac{Pd}{W\lambda} = \frac{\tan\theta - \theta}{\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

where $\theta = n\lambda$. The couple C is accordingly $(W\lambda\tan\theta)/\theta$.

The question naturally presents itself as to what happens to these results when P is withdrawn. As P tends to zero so does n, and θ with it. It is known that the indeterminate $(\tan\theta)/\theta$ tends to unity, so that C takes the value $W\lambda$ which we expect. The first few terms in the expansion give

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \ldots,$$

so that, to the lowest order of small quantities, $(\tan\theta - \theta)/\theta$ approaches $\theta^2/3$, or $(n\lambda)^2/3$. On replacing P by n^2EI the equation (iv) for the deflection becomes

$$\frac{n^2EId}{W\lambda} \to \frac{(n\lambda)^2}{3},$$

so that d takes the value $W\lambda^3/3EI$. This accords with the result given in texts on structures for the end-deflection of a simply loaded cantilever. It will be noticed that if d is the deflection when P is present and d_1 when absent, the ratio d/d_1 can be written $3(\tan\theta - \theta)/\theta^3$. The rather surprising result follows that for a given P the proportional change in the deflection, being governed by θ alone, is independent of W.

One further observation is worth making. If P be reversed, so that the beam acts as a tie instead of a strut, the trigonometrical functions in the complementary function are replaced by hyperbolic functions. This is exemplified in the following more complicated example. As the method has already been explained, part of the routine analysis is left to be supplied by the reader.

A horizontal beam of length λ is encastrée at both ends and acts as a tie. There is end-pull P and uniform load w per unit length. The total load $w\lambda$ is therefore supported by two reactions of $\frac{1}{2}w\lambda$ and the equation of equilibrium is

$$EIy'' = C + Py - \tfrac{1}{2}w\lambda x + \tfrac{1}{2}wx^2.$$

On determining the complementary function and the particular integral we have

$$y = A\cosh nx + B\sinh nx - \frac{wx^2}{2P} + \frac{w\lambda x}{2P} - \left(\frac{C}{P} + \frac{w}{n^2P}\right),$$

where $n^2 = P/EI$. The end conditions at the left are $x = 0 = y = y'$, which gives

$$A = \frac{c}{P} + \frac{w}{n^2P}, \quad B = -\frac{w\lambda}{2nP}.$$

The solution thus becomes

$$n^2Py = (w + n^2C)(\cosh nx - 1) - \tfrac{1}{2}wn\lambda(\sinh nx - nx) - \tfrac{1}{2}wn^2x^2. \quad \text{(i)}$$

We may now either use the right end of the beam and say $x = \lambda$, $y = 0 = y'$; or we may trade on the symmetry of the configuration and say $x = \frac{1}{2}\lambda$, $y' = 0$. They all lead to the same result, which is

$$C = \frac{w\lambda^2}{4} \frac{\theta \coth \theta - 1}{\theta^2},$$

where $\theta = \frac{1}{2}n\lambda$. The central deflection, on using the value of C and putting $x = \frac{1}{2}\lambda$, $y = d$ in (i), is given by

$$n^2Pd = w\theta(\tfrac{1}{2}\theta - \tanh \tfrac{1}{2}\theta)$$

or
$$\frac{Pd}{w\lambda^2} = \frac{\frac{1}{2}\theta - \tanh \frac{1}{2}\theta}{4\theta}.$$

3, 9·1. *Critical Thrust.*

When the lateral loading is absent and the only lateral forces are such as ensure the maintenance of the configuration, the problem becomes one of stability. The function of a strut is to withstand compression; but it is common knowledge that a long or thin strut is likely to buckle. A long lath or T-square will illustrate the point. The load which causes buckling is variously known as the buckling, crippling, or critical load. Apart from such factors as would affect the strength of any member whatever, the buckling load is dependent on the methods of fixing the ends. Its determination frequently depends on the solution of transcendental equations. The following case is particularly simple.

A horizontal strut of length λ with both ends pinned has end-thrust P. Assuming that it curves downwards, there is statical equilibrium and no lateral retaining forces are required. In conformity with previous work, we write down

$$EIy'' = C - Py.$$

As the ends are merely pinned there are no end-couples, and C is zero. There is no particular integral, and the solution consists solely of the complementary function $y = R \sin(nx + \phi)$. With the end conditions $x = 0 = y$, we have $R \sin \phi$ zero. The possibility $R = 0$ must be ruled out as it leaves us with no solution. The alternative is $\phi = 0$, and we have $y = R \sin nx$. The conditions at the right are $x = \lambda$, $y = 0$, so that $R \sin n\lambda$ is zero, whence $n\lambda = 0, \pi, 2\pi$, &c. We can discard the zero, and the smallest value that is likely to give trouble is $n\lambda = \pi$, whence $P = EI(\pi/\lambda)^2$ as the critical thrust.

It is sometimes wondered why R remains indeterminate, so that the curved form of the beam cannot be found. The answer is that once buckling sets in the mischief is done, and matters go from bad to worse. It is true that a member can be held flexed under end-thrust, witness an ordinary bow for shooting arrows; but for this a more delicate analysis is required.

Turning now to a more complicated case, consider the problem when the same strut has the left end clamped horizontally and the right end pinned, there being horizontal thrust P. There is now a counterclockwise couple C at the left, and if the right end is to preserve alignment and suffer no deflection, there must be a downward restraining force R. Statics dictates that an equal and opposite force R must act upward at the wall, and accordingly we have $C = R\lambda$. The equation of equilibrium is

$$EIy'' = C - Py - Rx,$$

of which the solution is

$$y = A\cos nx + B\sin nx + \frac{C - Rx}{P}.$$

The arbitrary constants are determined from the conditions $x = 0 = y = y'$ at the left, whence

$$A = -\frac{C}{P}, \quad B = \frac{R}{nP}.$$

The solution becomes

$$y = \frac{C}{P}(1 - \cos nx) + \frac{R}{nP}(\sin nx - nx).$$

At the right end there is no couple, so that $y'' = 0$ when $x = \lambda$. On applying this to the equilibrium equation we get $C = R\lambda$, already known from statics and a check on our working. There is no deflection at the right, or $y = 0$ when $x = \lambda$. This leads to $\tan\theta = \theta$, where $\theta = n\lambda$. We derive the solution of this transcendental equation by regarding it as the intersection of the two simple graphs $y = \theta$ and $y = \tan\theta$. Notice that both of these pass through the origin at 45° to OX. A rough sketch indicates that there is an infinite number of intersections, and the horizontal interval between the higher values gradually tends to the constant value π. What is of more importance to us is that the first intersection falls slightly short of $3\pi/2$, or 4·71.

3, 9·2. *Newton's Method.*

In order to improve on this approximation we employ Newton's method. Assuming that α is a good approximation to the root of an equation $f(x) = 0$, we have $f(\alpha)$ close to zero. If the correct root is $\alpha + h$, then h is small (and is known as the correction) and $f(\alpha + h)$ is precisely zero. As $f(\alpha + h)$ differs but little from $f(\alpha) + hf'(\alpha)$ we have pretty well $f(\alpha) + hf'(\alpha) = 0$, whence $h = -f(\alpha)/f'(\alpha)$. The process can be repeated if a closer result is required.

Reverting to the transcendental equation for our strut, we have

$$\tan\theta - \theta = 0 = f(\theta)$$

and
$$f'(\theta) = \sec^2\theta - 1 = \tan^2\theta.$$

We take as our approximate root, say $\theta = 4{\cdot}5$ radians (why not $3\pi/2$?), and deduce a correction of

$$h = -\frac{\tan 4{\cdot}5 - 4{\cdot}5}{\tan^2 4{\cdot}5} \simeq -0{\cdot}01.$$

This gives $\theta = 4{\cdot}49 = n\lambda$, whence the critical thrust is $P = 20{\cdot}2EI/\lambda^2$.

3, 9·3. The following particularly simple example is introduced to illustrate two important points. It frequently happens that the clear span of a beam is interrupted by intermediate loads. In such cases an equation of equilibrium is valid only in the interval for which it is formed, and it is illegitimate to determine the constants from conditions outside the interval. In awkward cases it may be necessary to have different equations in different intervals, in which case the solutions must tally at the common end-points. The tally consists in giving the same deflection and the same slope; for no sudden change of slope is possible unless the beam be broken. Such problems are usually rather tedious and involve a good deal of algebraic manipulation. The following is an exception.

A horizontal beam of length λ is simply supported at each end. There is central load W and end-thrust P. The reaction at each support is $\frac{1}{2}W$, and if we work in the left half-span the equation of equilibrium is

$$EIy'' + Py = -\tfrac{1}{2}Wx.$$

It is easily verified that if our imaginary section is in the right half-span we get the somewhat different equation

$$EIy'' + Py = \tfrac{1}{2}W(x - \lambda).$$

Adopting the former equation we have as the solution

$$y = A\cos nx + B\sin nx - \frac{Wx}{2P}.$$

The conditions at the left give $A = 0$, so that

$$y = B\sin nx - \frac{Wx}{2P}.$$

The symmetry of the configuration permits us to assume that $y' = 0$ when $x = \frac{1}{2}\lambda$. Hence

$$0 = nB\cos\tfrac{1}{2}n\lambda - \frac{W}{2P}, \quad B = \frac{W\sec\frac{1}{2}n\lambda}{2Pn}.$$

Note that, had we trespassed into the right half-span by using $x = \lambda$, $y = 0$, we should have reached the erroneous result $B = (W\lambda\operatorname{cosec} n\lambda)/2P$. The correct result leads to

$$\frac{2nPy}{W} = \sec\tfrac{1}{2}n\lambda\sin nx - nx,$$

which gives a central deflection of $W(\tan\frac{1}{2}n\lambda - \frac{1}{2}n\lambda)/2nP$. The maximum bending moment occurs at the middle, and the equilibrium equation gives its value as $(W\tan\frac{1}{2}n\lambda)/2n$.

The next point we wish to make is numerical, and the verification of the details is left to the reader. Suppose the beam is a steel shaft of diameter 4 in. and length 15 ft. An end-thrust of 5000 lb./in.^{2} is equivalent to $P = 6{\cdot}28 \,.\, 10^4$ lb. A central load which gives a maximum skin stress of 4000 lb./in.^{2} is equivalent to $W = 559$ lb. It will be observed that these stresses are respectively less than 3 and 2 tons/in.^{2}, and are accordingly quite safe. Further calculation gives $I = 12{\cdot}57$ in.^{4}, and if we give E the usual value $3 \,.\, 10^7$ lb./in.^{2}, we have

$$n = 1{\cdot}29 \,.\, 10^{-2}\ \text{in.}^{-1},$$

whence $n\lambda = 2{\cdot}33$ radians. The maximum bending moment, from $(W\tan\frac{1}{2}n\lambda)/2n$, now figures out at $5{\cdot}03 \,.\, 10^4$ in.-lb., and this corresponds to a maximum skin stress $8{\cdot}0 \,.\, 10^3$ lb./in.^{2} due to bending alone. Adding the $5 \,.\, 10^3$ lb./in.^{2} due to direction compression, we reach the total of $13{\cdot}0 \,.\, 10^3$ lb./in.^{2}, practically 6 tons, a figure not to be envisaged lightly. The moral is that it can be highly dangerous to load a strut laterally.

EXERCISES

1. If the beam be built in at both ends and has central load W with end-thrust P, prove (i) the end and central couples are each $(W\lambda \tan\theta)/8\theta$ where $\theta = n\lambda/4$; (ii) the two inflections occur at $\lambda/4$ from either end; (iii) the central deflection is $W\lambda(\tan\theta - \theta)/4P\theta$, and as P approaches zero it approaches the standard value $W\lambda^3/192EI$.

2. If in the previous problem the central load is replaced by a uniform load w per unit length and $\theta = \frac{1}{2}n\lambda$, prove that (i) the end-couples are $w\lambda^2(1 - \theta \cot\theta)/4\theta^2$; (ii) the central couple is $w\lambda^2(1 - \theta \operatorname{cosec}\theta)/4\theta^2$; (iii) the central deflection is $w\lambda^2(\tan\frac{1}{2}\theta - \frac{1}{2}\theta)/4P\theta$; the inflections are given by $\theta\cos(\theta - nx) = \sin\theta$. Verify that as P approaches zero the results are in conformity with the usual standards.

3. If a simple cantilever has end-thrust P and uniform load w per unit length, prove that with $\theta = n\lambda$, (i) the retaining couple is $w\lambda^2(1 - \sec\theta + \theta\tan\theta)/\theta^2$, which approaches $\frac{1}{2}w\lambda^2$ as P tends to zero; (ii) the end-deflection is

$$w\lambda^2(1 - \sec\theta + \theta\tan\theta - \tfrac{1}{2}\theta^2)/P\theta^2$$

and tends to $w\lambda^4/8EI$.

4. A horizontal tie is simply supported at each end. The pull is P and uniform load w per unit length. Prove that with $\theta = n\lambda/4$ the central deflection is $w\lambda^2(\theta^2 - \sinh\theta\tanh\theta)/8P\theta^2$, which tends to $w\lambda^4/48EI$ as P tends to zero.

5. A horizontal steel bar of length 3 metres has cross-section 2 cm. square and both ends clamped. If the end-thrust is 200 Kgm., prove that a central load of 8 Kgm. would give a deflection of 2 cm. and raise the maximum stress from 50 Kgm./cm.^{2} to 300 Kgm./cm.^{2}, if $E = 2 \cdot 10^6$ Kgm./cm.^{2}.

6. A cantilever of circular section has length equal to 80 diameters. Prove that, in the presence of an end-load, an end-thrust equivalent to 5 Kgm./cm.^{2} would change the deflection by about 11 per cent, if $E = 2 \cdot 10^6$ Kgm./cm.^{2}.

3, 10. $f(x) = B\sin\omega x + A\cos\omega x$.

We begin the study of this form by investigating the effect of the linear operator $F(D)$ on $R\sin(\omega x - \phi)$. We have

$$(D^2 + \omega^2)R\sin(\omega x - \phi) \equiv 0,$$

whence by successive differentiations the operators $(D^3 + \omega^2 D)$ and $(D^4 + \omega^2 D^2)$ both annihilate $R\sin(\omega x - \phi)$. This shows that the operator D^2 is equivalent to multiplication by $-\omega^2$, and if we write this as $D^2 = -\omega^2$, we have in succession

$$D^3 = -\omega^2 D, \quad D^4 = -\omega^2 D^2 = +\omega^4,$$

and so on. It follows that, apart from certain exceptions to be noted later, our operator $F(D)$ can be reduced to the equivalent form $(p + qD)$. In special cases, either or both of p and q may be zero.

Example 1.- Since $D^2 \sin 3x = -9 \sin 3x$, we have

$$(D^3 - D^2 + 13D - 6) \sin 3x = (-9D + 9 + 13D - 6) \sin 3x$$
$$= (3 + 4D) \sin 3x.$$

We accordingly turn our attention to the operator $(p + qD)$ and note that

$$(p + qD) \sin \omega x = p \sin \omega x + \omega q \cos \omega x$$
$$= k \sin (\omega x + \psi),$$

where

$$p = k \cos \psi, \qquad k^2 = p^2 + \omega^2 q^2,$$
$$\omega q = k \sin \psi, \quad \tan \psi = \omega q/p.$$

Similarly,

$$(p + qD)R \sin (\omega x - \phi) = kR \sin (\omega x - \phi + \psi).$$

These results are still valid if the sign of p or q be changed throughout.

Example 2.—As an illustration, we have

$$(5 - 2D) \sin 4x = 5 \sin 4x - 8 \cos 4x.$$

Putting
$$5 = k \cos \psi, \; -8 = k \sin \psi,$$
we see that ψ must lie in the fourth quadrant and be numerically equal to $\tan^{-1} 8/5$. Also $k^2 = 8^2 + 5^2 = 89$, although in practice it is quicker (having first determined ψ) to determine k as $8 \operatorname{cosec} \psi$. We have the result

$$(5 - 2D) \sin 4x = \sqrt{89} \sin (4x - \psi), \; \tan \psi = 8/5.$$

This might have been written straight down according to rule.

It appears from this that if the equation has the form

$$F(D)y = B \sin \omega x, \qquad \dots\dots \quad \text{(i)}$$

it is a reasonable assumption that y has the form $R \sin (\omega x - \phi)$. After substitution we may hope to make the results tally by adjustment of the constants. If $F(D)$ reduces to $(p + qD)$, we have (i) in fact replaced by

$$B \sin \omega x = (p + qD)R \sin (\omega x - \phi)$$
$$= kR \sin (\omega x - \phi + \psi).$$

Hence $R = B/k$, $\phi = \psi$, and

$$y = \frac{B}{\sqrt{(p^2 + \omega^2 q^2)}} \sin (\omega x - \psi), \; \tan \psi = \frac{\omega q}{p}.$$

Example 3.—To determine the particular integral of

$$(D^3 - D^2 + 13D - 6)y = 7 \sin 3x,$$

we assume $y = R\sin(3x - \varphi)$. We then have

$$\begin{aligned} 7\sin 3x &= (D^3 - D^2 + 13D - 6)R\sin(3x - \varphi) \\ &= (3 + 4D)R\sin(3x - \varphi) \\ &= \sqrt{(153)}R\sin(3x - \varphi + \psi),\ \tan\psi = 4. \end{aligned}$$

Hence $R = 7/\sqrt{(153)}$ and $\varphi = \tan^{-1}4$, so that

$$y = \frac{7}{\sqrt{(153)}}\sin(3x - \varphi).$$

3, 11. *Symbolic Method.*

If we treat the above Example 3 symbolically, ignoring for the moment the fact that D is really d/dx, and not bothering whether what we are doing has any meaning, we might write

$$y = \frac{7}{D^3 - D^2 + 13D - 6}\sin 3x = \frac{7}{3 + 4D}\sin 3x$$

after replacing D^2 by -9. Borrowing an idea from the theory of surds we might now multiply numerator and denominator by the co-factor $(3 - 4D)$. The denominator would become $(9 - 16D^2)$, or 153 after replacing D^2 by -9. This gives

$$y = \frac{7(3 - 4D)}{(3 + 4D)(3 - 4D)}\sin 3x = \frac{7}{153}(3 - 4D)\sin 3x.$$

As $$(3 - 4D)\sin 3x = \sqrt{(153)}\sin(3x - \phi),\ \tan\phi = 4,$$

we reach $$y = \frac{7}{\sqrt{(153)}}\sin(3x - \phi)$$

as before. In general, if $F(D)y = B\sin\omega x$, we could write

$$\begin{aligned} y = \frac{B}{F(D)}\sin\omega x &= \frac{B}{p + qD}\sin\omega x \\ &= \frac{B(p - qD)}{p^2 + \omega^2 q^2}\sin\omega x \\ &= \frac{B}{\sqrt{(p^2 + \omega^2 q^2)}}\sin(\omega x - \psi),\ \tan\psi = \frac{\omega q}{p} \end{aligned}$$

as before. The symbolic method justifies itself in giving the correct result in cases which can be checked by other means. A much wider use is made of symbolic methods in Heaviside's *Operational Calculus*; the justification is not always simple.

The two working rules are:

(i) $(p + qD)\sin\omega x = \sqrt{(p^2 + \omega^2 q^2)}\sin(\omega x + \psi), \tan\psi = \omega q/p,$

(ii) $$\frac{1}{p + qD}\sin\omega x = \frac{1}{\sqrt{(p^2 + \omega^2 q^2)}}\sin(\omega x - \psi).$$

The rule is easy to remember since the square root, which is the same in both cases, is above or below according as the operator is above or below. Similarly, the phase angle is correspondingly added or subtracted. It is immaterial if ωx is replaced by $(\omega x + \phi)$; and in virtue of the relation $\cos\omega x = \sin(\omega x + \frac{1}{2}\pi)$ the rules apply equally well to cosines.

It may happen that the substitution $D^2 = -\omega^2$ reduces $F(D)$ to a mere multiple of D, in which case the equation 3, 10 (i) reads $Dy = B\sin\omega x$. Direct integration then gives $y = -(B/\omega)\cos\omega x$. If the symbolic method is employed, the equation reads

$$y = \frac{1}{D}B\sin\omega x.$$

Hence $1/D$ must be interpreted quite literally as the inverse of a differentiation, i.e. an integration.

Example 1.—$(D^3 - D^2 + 13D - 9)y = 8\cos(3x + \alpha)$ gives

$$y = \frac{8}{D^3 - D^2 + 13D - 9}\cos(3x + \alpha)$$

$$= \frac{8}{4D}\cos(3x + \alpha) = \tfrac{2}{3}\sin(3x + \alpha),$$

a result that can be verified by substitution.

When the equation has the more complete form

$$F(D)y = A\cos\omega x + B\sin\omega x, \quad \text{(i)}$$

it is now apparent that the particular integral will have the form

$$y = H\cos\omega x + K\sin\omega x. \quad \text{(ii)}$$

There are two courses open to us. We begin by reducing the operator, replacing D^2 by $-\omega^2$ wherever possible. The equation then becomes

$$(p + qD)y = A\cos\omega x + B\sin\omega x. \quad \text{(iii)}$$

We may either (*a*) make the substitution (ii) in (iii) and determine

H, K by comparison of coefficients; or (*b*) express the right side of (iii) with an amplitude and a phase angle, then apply rule (ii) from 3, 11. If we adopt the former course, we have

$$A = pH + \omega q K, \quad B = pK - \omega q H$$

for the determination of H, K. If we pursue the second course and put $A \cos \omega x + B \sin \omega x = R \sin(\omega x + \alpha)$, $R = \sqrt{(A^2 + B^2)}$, $\tan \alpha = A/B$,

we have $y = \dfrac{1}{p + qD} R \sin(\omega x + \alpha)$

$$= \left(\frac{A^2 + B^2}{p^2 + \omega^2 q^2}\right)^{1/2} \sin(\omega x + \alpha - \phi), \quad \tan \phi = \frac{\omega q}{p}.$$

The two results will, of course, merely be two ways of writing the same expression.

Example 2.—$(D^3 - D^2 + 5D - 6)y = 21 \sin 2x + 20 \cos 2x$. On reducing the operator by $D^2 = -4$, we have

$$(D - 2)y = 21 \sin 2x + 20 \cos 2x = 29 \sin(2x + \alpha), \ \tan \alpha = 20/21.$$

(*a*) Putting $\qquad y = H \cos 2x + K \sin 2x,$

we have $\qquad -2H - 2K = 21, \quad -2H + 2K = 20,$

whence $\qquad H = -41/4, \quad K = -1/4.$

Hence $\qquad y = -(\sin 2x + 41 \cos 2x)/4.$

(*b*) From $\qquad (D - 2)y = 29 \sin(2x + \alpha)$

we have
$$y = -\frac{29}{2 - D} \sin(2x + \alpha)$$

$$= -\frac{29}{2\sqrt{2}} \sin(2x + \alpha + \varphi), \ \tan \varphi = 1.$$

As $\tan(\alpha + \varphi) = 41$ and $1^2 + 41^2 = 2 . 29^2$, the two results agree.

3, 12. *Exceptional Case.*

An exception occurs when the substitution $D^2 = -\omega^2$ makes $F(D)$ identically zero. The usual procedure is then nugatory. This occurs when $(D^2 + \omega^2)$ is a factor of $F(D)$; incidentally $(A \cos \omega x + B \sin \omega x)$ is then part of the complementary function. We can resolve the difficulty by studying the properties of the function $x \sin \omega x$ and its companion $x \cos \omega x$. We have

$$D(x \sin \omega x) = x(D \sin \omega x) + \sin \omega x (Dx)$$
$$= \omega x \cos \omega x + \sin \omega x.$$

Similarly, $D^2(x \sin \omega x) = x(D^2 \sin \omega x) + 2(Dx)(D \sin \omega x)$

$$= -\omega^2 x \sin \omega x + 2\omega \cos \omega x.$$

In general, Leibnitz' theorem on the differentiation of a product gives

$$D^n(x \sin \omega x) = x(D^n \sin \omega x) + nD^{n-1} \sin \omega x.$$

If we use this result for various values of n, we have

$$F(D)x \sin \omega x = xF(D) \sin \omega x + F'(D) \sin \omega x.$$

The term $F(D) \sin \omega x$ is known to be zero by hypothesis. The other term $F'(D)$ will give a result of the form $(a \cos \omega x + b \sin \omega x)$. Similarly, $F(D)x \cos \omega x$ would give $(c \cos wx + d \sin \omega x)$. It appears from this that the assumption

$$y = Hx \cos \omega x + Kx \sin \omega x \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

could, by adjustment of constants, be made to fit the failure case of

$$f(x) = A \cos \omega x + B \sin \omega x$$

by equating to $F'(D)(H \cos \omega x + K \sin \omega x)$.

Example.—To find the particular integral of

$$(D^3 - 2D^2 + 9D - 18)y = 6 \cos 3x + 48 \sin 3x,$$

we note that the substitution $D^2 = -9$ reduces the operator to zero: in fact $(D^2 + 9)$ is a factor. We accordingly make the assumption (i). The reader is advised to work out the effect of the substitution in detail and compare it with the following procedure.

$$F(D) = D^3 - 2D^2 + 9D - 18 \to 0$$
$$F'(D) = 3D^2 - 4D + 9 \to -2(9 + 2D).$$

Hence the substitution (i) gives

$$F(D)y = F'(D)(H \cos 3x + K \sin 3x)$$
$$= 6[(2H - 3K) \sin 3x - (3H + 2K) \cos 3x].$$

Comparison gives $\quad 2H - 3K = 8, \; 3H + 2K = -1,$

whence $H = 1$, $K = -2$. The required particular integral is

$$y = x(\cos 3x - 2 \sin 3x).$$

The most important application in practice is to find the particular integral in the failure case from

$$(D^2 + \omega^2)y = R \sin(\omega x + \phi).$$

We assume

$$y = Hx \sin(\omega x + \psi),$$

so that
$$F(D)y = F(D)Hx\sin(\omega x + \psi),\ F(D) = D^2 + \omega^2,$$
$$= HxF(D)\sin(\omega x + \psi) + HF'(D)\sin(\omega x + \psi)$$
$$= 2HD\sin(\omega x + \psi) = 2\omega H\cos(\omega x + \psi),$$
$$R\sin(\omega x + \phi) = 2\omega H\sin(\omega x + \psi + \tfrac{1}{2}\pi).$$

Hence, by comparison, we have
$$2\omega H = R,\ \psi + \tfrac{1}{2}\pi = \phi,$$
so that
$$y = \frac{R}{2\omega}x\sin(\omega x + \phi - \tfrac{1}{2}\pi) = -\frac{R}{2\omega}x\cos(\omega x + \phi).$$

A symbolic method of obtaining these results will be given later; but the method sometimes given, of using a limiting process, is not to be recommended.

EXERCISES

1. $y'' - 8y' + 15y = 3\sin 2x.$ $\left[y = Ae^{3x} + Be^{5x} + \frac{3}{\sqrt{(377)}}\sin(2x + \varphi).\right]$

2. $(D^2 + 13)y = 3\cos 4x.$ $[y = R\sin(x\sqrt{13} + \varphi) - \cos 4x.]$

3. $(D^2 + 2D + 9)y = \cos 3x.$ $[y = Re^{-x}\sin(2x\sqrt{2} + \varphi) + \frac{1}{6}\sin 3x.]$

4. Find the particular integral of
$$3y - 2y' + 2y'' = \sin 3x.$$
Verify your answer by substitution.

5. Find by both methods the particular integral of
$$(D^3 - D^2 + 4D + 3)y = 8\sin x + 15\cos x$$
and verify that they agree.

6. Find the particular integral of
$$\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} - 4y = 5\sin 2x. \qquad [y = \tfrac{1}{4}x\cos 2x.]$$

7. What is the particular integral of the equation
$$y''' - 2y'' + y' - 11y = 5\sin 2x - 5\cos 2x.$$
$[y = \cos 2x + \frac{1}{3}\sin 2x.]$

8. Examine how far the methods of the text can be applied when $f(x)$ has the form $A\cosh\omega x + B\sinh\omega x$. Apart from replacing D^2 by ω^2, what other modifications are necessary?

9. Solve the equation $m\dfrac{d^2z}{dt^2} + F\dfrac{dz}{dt} = qE\sin\omega t$ which occurs in the theory of the ionized layer. Find A, B on the assumption that it has a solution of the form
$$z = A\cos\omega t + B\sin\omega t.$$

3, 13. $f(x) = e^{ax}X$.

The key to this form is that the first and all succeeding differential coefficients of e^{ax} are multiples of e^{ax}. We thus have

$$D(e^{ax}V) = e^{ax}(DV) + (De^{ax})V = e^{ax}(D + a)V.$$

Similarly,

$$D^2(e^{ax}V) = e^{ax}(D^2V) + 2(De^{ax})(DV) + (D^2e^{ax})V$$
$$= e^{ax}(D + a)^2V,$$

and in general by Leibnitz' theorem

$$D^n(e^{ax}V) = e^{ax}(D^nV) + n(De^{ax})(D^{n-1}V) + \ldots + (D^ne^{ax})V$$
$$= e^{ax}(D + a)^nV.$$

Hence

$$F(D)\{e^{ax}V\} = (a + bD + cD^2 + \ldots)(e^{ax}V)$$
$$= e^{ax}\{a + b(D + a) + c(D + a)^2 + \ldots\}V$$
$$= e^{ax}F(D + a)V,$$

i.e. we can transpose e^{ax} and replace D by $(D + a)$.

It appears that if the equation has the form

$$F(D)y = e^{ax}X$$

we might expect to get a particular integral by assuming $y = e^{ax}V$, where X and V are of the same form, i.e. both polynomial or both trigonometrical.

Example 1.—$(D^2 + 6D + 5)y = 3e^x \sin 2x$.

Since $(D + 1)^2 + 6(D + 1) + 5 \equiv D^2 + 8D + 12$,

the assumption $y = e^x(H \sin 2x + K \cos 2x)$

gives

$$(D^2 + 6D + 5)\{e^x(H \sin 2x + K \cos 2x)\}$$
$$= e^x(D^2 + 8D + 12)(H \sin 2x + K \cos 2x)$$
$$= e^x(8D + 8)(H \sin 2x + K \cos 2x)$$
$$= 8e^x\{(H - 2K) \sin 2x + (2H + K) \cos 2x\}.$$

Equating the corresponding parts, we have

$$8(H - 2K) = 3, \quad H = \frac{3}{40},$$

$$8(2H + K) = 0, \quad K = -\frac{6}{40}.$$

The particular integral is thus

$$y = \frac{3}{40} e^x(\sin 2x - 2\cos 2x).$$

The result might have been reached symbolically. We can write the equation as

$$\begin{aligned} y &= \frac{3}{D^2 + 6D + 5} e^x \sin 2x \\ &= e^x \frac{3}{D^2 + 8D + 12} \sin 2x \\ &= \tfrac{3}{8} e^x \frac{1}{1 + D} \sin 2x \\ &= \frac{3}{8\sqrt{5}} e^x \sin(2x - \varphi), \quad \tan\varphi = 2. \end{aligned}$$

This is equivalent to the previous form.

3, 14. In the simple case where X is a constant the equation has the form

$$F(D)y = Ae^{ax}.$$

On the grounds that $D^3e^{ax} = a^3e^{ax}$ with similar results for other powers of D, we have

$$F(D)e^{ax} = F(a)e^{ax}.$$

Hence the assumption $y = Ce^{ax}$ gives

$$F(D)y = F(D)Ce^{ax} = CF(a)e^{ax} = Ae^{ax}.$$

This determines C, and we have

$$y = \frac{A}{F(a)} e^{ax}$$

as the particular integral. The result might have been obtained symbolically. We should write the equation as

$$y = \frac{A}{F(D)} e^{ax} = e^{ax} \frac{A}{F(D + a)} \cdot 1.$$

We should then ignore D as acting on a constant and so reach the same result as before.

The foregoing result is useless if $F(a)$ is zero. This must occur if $(D - a)$ is a factor of $F(D)$, in which case e^{ax} is part of the complementary function. The equation could then be written as

$$\{\psi(D)\}(D - a)y = Ae^{ax}.$$

By writing $\psi(D)y = u$ we get $(D - a)u = Ae^{ax}$. This case has already been investigated in connexion with repeated roots in the complementary function, see 3, 2. The result is known to be $u = Axe^{ax}$. The equation thus reduces to the form

$$\psi(D)y = Axe^{ax}$$

and the proper assumption would be $y = (p + qx)e^{ax}$. Substitution gives

$$\begin{aligned}\psi(D)y &= \psi(D)\{(p + qx)e^{ax}\} \\ &= e^{ax}\{\psi(D + a)\}(p + qx) \\ &= Axe^{ax}.\end{aligned}$$

A comparison would, in any particular case, determine the coefficients. It is left to the reader, using the expansion of $\psi(D + a)$, to show that the coefficient of x in $\{\psi(D + a)\}(p + qx)$ is $q\psi(a)$, which must accordingly equal A. Any other terms that occur are multiples of e^{ax} and can be discarded into the complementary function; our particular integral is

$$y = \frac{Axe^{ax}}{\psi(a)}.$$

The same result can be obtained more simply. Presuming that $\psi(D)$ does not contain the factor $(D - a)$, then

$$u = \frac{Ae^{ax}}{\psi(a)}$$

is certainly a solution of $\quad \psi(D)u = Ae^{ax}$

as substitution shows. Putting $u = (D - a)y$, we conclude that

$$F(D)y = (D - a)\{\psi(D)\}y = Ae^{ax}$$

is satisfied if

$$(D - a)y = \frac{Ae^{ax}}{\psi(a)}.$$

As pointed out above, the solution of this is known to be

$$y = \frac{Axe^{ax}}{\psi(a)}.$$

EXERCISES

1. $(D^2 - D - 2)y = e^x$. $\quad [y = Ae^{2x} + Be^{-x} - \frac{1}{2}e^x.]$
2. $(D^2 - D - 2)y = e^{2x}$. $\quad [y = Ae^{2x} + Be^{-x} + \frac{1}{3}xe^{2x}.]$
3. $(D^2 + D)y = \cosh x$. $\quad [y = \frac{1}{4}e^x - \frac{1}{2}xe^{-x} + Ae^{-x} + B.]$

4. $(D^3 + D + 2)y = e^{-x}\cos 2x$.

$$\left[y = Ae^{-x} + \frac{e^{-x}}{12}\cos 2x + Re^{\frac{1}{2}x}\sin(\tfrac{1}{2}x\sqrt{7} + \varphi).\right]$$

5. Find the particular integral of

$$(D^3 - 3D + 7)y = 5x^2e^x. \qquad [y = (x^2 - \tfrac{6}{5})e^x.]$$

6. By the use of the substitution $y = ue^{\alpha x}$, prove that the equation

$$(D - \alpha)^2 y = Ce^{\alpha x}$$

is satisfied if $u = \frac{1}{2}Cx^2$. Deduce that if $F(D)$ has the repeated factor $(D - \alpha)^2$, the equation

$$F(D)y = (D - \alpha)^2\{\psi(D)\}y = Ae^{\alpha x}$$

has the particular integral $\frac{1}{2}Ax^2e^{\alpha x}/\psi(\alpha)$.

Apply the method to find the particular integral of

$$(D^3 - D^2 - D + 1)y = 2e^x,$$

and verify the answer by substitution. Generalize the result.

7. Find the particular integral of the equation

$$(D^2 - 3D + 2)y = e^x(1 - 2x). \qquad [y = e^x(x + x^2).]$$

8. How is the particular integral of the equation $F(D)y = e^{\alpha x}\cos\omega x$ to be found if $(D^2 + \omega^2)$ is a factor of $F(D + \alpha)$?

Find the particular integral of

$$(D^3 + 5D^2 + 17D + 13)y = e^{-2x}\cos 3x.$$

$$\left[y = -\frac{xe^{-2x}}{20}(\cos 3x + 3\sin 3x)\right].$$

3, 15. *Symbolic Method.*

An uncritical use of the symbolic method is capable of giving the particular integral when $f(x)$ is a polynomial. The procedure consists in expanding $1/F(D)$ in ascending powers of D by the binomial theorem, possibly in conjunction with partial fractions. If the result is

$$\frac{1}{F(D)} = \alpha + \beta D + \gamma D^2 + \ldots$$

the equation $F(D)y = f(x)$ is written

$$y = \frac{1}{F(D)}f(x) = (\alpha + \beta D + \gamma D^2 \ldots)(L + Mx + Nx^2 + \ldots).$$

Evidently the expansion of $1/F(D)$ need not be carried above the degree of $f(x)$. Taking the previous 3, 8, Example 1, let

$$(3 + 2D - D^2 - D^3)y = 9 - 2x + 3x^2.$$

We write
$$\frac{1}{F(D)}=\frac{1}{3}\left\{1+D\left(\frac{2}{3}-\frac{D}{3}\right)\right\}^{-1}$$
since powers above the second are not required. This gives
$$\frac{1}{3}\left\{1-D\left(\frac{2}{3}-\frac{D}{3}\right)+D^2\left(\frac{2}{3}-\frac{D}{3}\right)^2-\ldots\right\}=\frac{1}{3}-\frac{2D}{9}+\frac{7D^2}{27}\ldots.$$
Hence
$$y=\frac{1}{3+2D-D^2-D^3}(9-2x+3x^2)$$
$$=\left(\frac{1}{3}-\frac{2D}{9}+\frac{7D^2}{27}\right)(9-2x+3x^2)$$
$$=5-2x+x^2$$
as before.

In redemption of a promise (see 3, 12), we reconsider the equation
$$(D^2+\omega^2)y=\sin\omega x.$$
We regard $i\sin\omega x$ as the imaginary part of $e^{i\omega x}$, and write
$$\frac{1}{(D-i\omega)(D+i\omega)}e^{i\omega x}=\frac{1}{D-i\omega}\left\{\frac{1}{D+i\omega}e^{i\omega x}\right\}$$
$$=\frac{1}{D-i\omega}\frac{e^{i\omega x}}{2i\omega}$$
$$=e^{i\omega x}\frac{1}{D}\frac{1}{2i\omega}=\frac{xe^{i\omega x}}{2i\omega}.$$
The imaginary part gives
$$y=-\frac{1}{2\omega}x\cos\omega x.$$

EXERCISES

1. Apply the symbolic method to the failure case of $F(D)y=Ae^{\alpha x}$ where $F(\alpha)=0$.

2. Apply the symbolic method to find the particular integral in the following cases already solved by other means:

(i) $D^3(D-3)y=9x+15$.
(ii) $D^2(3D+2)y=12x^2+2$.
(iii) $(D^2+5D+6)y=2x+4$.
(iv) $(D^2-3D+2)y=e^x(1-2x)$.
(v) $(D^4+3D^2-4)y=5\sin 2x$.

3. Use the symbolic method to deduce

$$\text{(i)} \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).$$

$$\text{(ii)} \int x^2 e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2).$$

$$\text{(iii)} \int x^2\cos 2x\,dx = \tfrac{1}{2}(x^2 - \tfrac{1}{2})\sin 2x + \tfrac{1}{2}x\cos 2x.$$

3, 16. *Applications.*

A good exemplification of the foregoing principles is provided by simple series circuits with an alternating voltage. We take the resistance to be R, the inductance L, capacity C, and the alternating voltage to be $E_0 \sin \omega t$. Beginning with the simple case where the capacity is negligible, we have

$$RI + L\frac{dI}{dt} = E_0 \sin\omega t = (R + LD)I.$$

This has already been discussed as a linear equation of the first order in 2, 19. Using the present methods, we have the complementary function given by $D = -R/L$, whence $I = A\exp(-Rt/L)$. This is the transient which rapidly dies out with increasing t. For the particular integral, we have

$$I = \frac{E_0}{R + LD}\sin\omega t = \frac{E_0\sin(\omega t - \phi)}{\sqrt{(R^2 + \omega^2L^2)}}, \quad \tan\phi = \frac{\omega L}{R}.$$

The lag and impedance have already been mentioned.

When the capacity is of some account and the resistance negligible, Ohm's law takes the form

$$RI = 0 = E_0\sin\omega t - V - L\frac{dI}{dt},$$

where V is the voltage across the capacity. As $Q = CV$ and

$$I = \frac{dQ}{dt} = C\frac{dV}{dt},$$

we have
$$CL\frac{d^2V}{dt^2} + V = E_0\sin\omega t.$$

Most mechanical systems when slightly disturbed from a position of stable equilibrium are capable of making small oscillations. The frequencies of these oscillations are part of the dynamics of the system and are known as "free" frequencies. The system may also be compelled to vibrate by external agency; such vibrations are said to be "forced", and their frequency depends on the disturbing agent. As an example, the wind-screen of a car can often be seen making forced vibrations when the car is stationary with the engine running.

The last equation shows that, apart from the disturbing agent $E_0 \sin \omega t$, the system can oscillate in accordance with the complementary function. Replacing CL by $1/n^2$ this gives

$$V = R \sin (nt + \phi),$$

where R, ϕ are arbitrary. The particular integral from

$$(D^2 + n^2)V = n^2 E_0 \sin \omega t$$

is
$$V = \frac{n^2 E_0}{D^2 + n^2} \sin \omega t = \frac{n^2 E_0 \sin \omega t}{n^2 - \omega^2} = \frac{E_0 \sin \omega t}{1 - CL\omega^2}.$$

A critical case occurs when the forcing rate nears the free rate; ω approaches n and the denominator tends to zero. The solution then becomes
$$V = -\tfrac{1}{2} nt E_0 \cos nt.$$

The presence of $\cos nt$ is a guarantee that the phenomenon is periodic; but the factor t shows that the amplitude tends to increase with time. It is erroneous to say the amplitude tends to infinity. For one thing the problem is idealized by the absence of resistance; and in nearly all similar cases the differential equation is only an approximation to the truth, based on the assumption that the displacements from equilibrium are small, the sort of approximation that replaces $\sin \theta$ by θ for small values. The tendency for the amplitude to become large when the forcing frequency approaches the free frequency is known as "resonance". It can be very dangerous, as in the case of a live load on a structure; on the other hand, it can be invaluable, as in wireless telegraphy.

The resonance case can be otherwise treated as follows. The complete solution is

$$V = A \cos nt + B \sin nt + \frac{n^2 E_0 \sin \omega t}{n^2 - \omega^2}, \quad n^2 = \frac{1}{CL}.$$

We can borrow part of the complementary function and write the particular integral as

$$\frac{n^2 E_0}{n^2 - \omega^2} (\sin \omega t - \sin nt) = -\frac{n^2 t\, E_0 \sin \frac{1}{2} t(\omega - n) \cos \frac{1}{2} t(\omega + n)}{\frac{1}{2} t(\omega - n)(\omega + n)}.$$

As ω approaches n the fraction $\{\sin\frac{1}{2}t(\omega - n)\}/\{\frac{1}{2}t(\omega - n)\}$ approaches unity. Thus the particular integral becomes $-\frac{1}{2}ntE_0 \cos nt$.

When all restrictions on the circuit are removed, Ohm's law gives

$$RI = E_0 \sin \omega t - V - L\frac{dI}{dt}$$

or

$$CL\frac{d^2V}{dt^2} + CR\frac{dV}{dt} + V = E_0 \sin \omega t.$$

We can write this as

$$(D^2 + 2bD + n^2)V = A \sin \omega t,$$

where

$$2b = \frac{R}{L}, \quad n^2 = \frac{1}{CL}, \quad A = \frac{E_0}{CL}.$$

Provided b is not too great, the complementary function gives the free but damped oscillations

$$V = Re^{-bt} \sin(ct + \phi), \; c^2 = n^2 - b^2.$$

The steady state is given by the particular integral

$$V = \frac{A}{D^2 + 2bD + n^2} \sin \omega t$$

$$= \frac{A}{(n^2 - \omega^2) + 2bD} \sin \omega t$$

$$= \frac{A \sin(\omega t - \psi)}{\sqrt{\{(n^2 - \omega^2)^2 + 4b^2\omega^2\}}}, \quad \tan \psi = \frac{2b\omega}{n^2 - \omega^2}.$$

A number of peculiarities can now be studied. Suppose firstly that the forcing frequency is very high, so that ω is large. The square root in the denominator is then dominated by the term $(\omega^2)^2$, so that the amplitude tends to A/ω^2, or $E_0/\omega^2 CL$, and tends to be very small. The value of $\tan \psi$ likewise tends to be small and negative, approaching the value $-2b/\omega$. The angle ψ accordingly approaches π, so that the phase is nearly opposite to the applied voltage.

If, alternatively, we suppose that ω approaches n, the term $(n^2 - \omega^2)^2$ tends to zero, and the amplitude approaches $A/2bn$ or $E_0\sqrt{L}/(R\sqrt{C})$. This may be large, but it shows no tendency to become infinite. It will appear in a moment that it is not even the largest possible amplitude, in spite of the apparent resonance. As for the phase angle, we see that $\tan \psi$ tends to infinity, so that ψ approaches $\frac{1}{2}\pi$.

In order to derive the maximum amplitude, we need the quantity under the surd in the denominator to be a minimum. We accordingly equate to zero its differential coefficient with respect to ω^2, which gives

$$-2(n^2 - \omega^2) + 4b^2 = 0.$$

It is instructive to write this in the form

$$n^2 - b^2 = c^2 = b^2 + \omega^2,$$

showing that $n > c > \omega$. This means that the forcing frequency $\omega/2\pi$ must be less than the free frequency $n/2\pi$ if the amplitude is to be a maximum. This maximum amplitude can be written

$$\frac{A}{2bc} = \frac{\frac{E_0}{R}\sqrt{\left(\frac{L}{C}\right)}}{\sqrt{\left(1 - \frac{R^2C}{4L}\right)}}.$$

If we differentiate the voltage equation and multiply by C, we get the equation for the current as

$$(CLD^2 + CRD + 1)I = CE_0 D \sin \omega t = CE_0 \omega \sin(\omega t + \tfrac{1}{2}\pi).$$

The steady current, obtained from the particular integral, is

$$I = \frac{CE_0\omega}{CLD^2 + CRD + 1} \sin(\omega t + \tfrac{1}{2}\pi)$$

$$= \frac{CE_0\omega}{(1 - CL\omega^2) + CRD} \sin(\omega t + \tfrac{1}{2}\pi)$$

$$= \frac{CE_0\omega \sin(\omega t + \tfrac{1}{2}\pi - \phi)}{\sqrt{\{(1 - CL\omega^2)^2 + C^2R^2\omega^2\}}}, \quad \tan\phi = \frac{CR\omega}{1 - CL\omega^2}.$$

The angle $(\omega t + \frac{1}{2}\pi - \phi)$ shows that the current is in phase with the applied voltage if $\phi = \frac{1}{2}\pi$. In that case $\tan\phi$ is infinite, so that $CL\omega^2 = 1$. This determines the correct frequency. It further happens that the square root simplifies, and we get

$$I = \frac{E_0}{R} \sin \omega t.$$

The effects of the inductance and the capacity have neutralized each other.

We will now briefly consider the corresponding mechanical problem. Suppose a vertical spring of stiffness k carries a mass m at the lower end. When x is the vertically downward displacement of m from the rest position, let $p\dot{x}$ be the damping. Let us suppose the upper end of the spring is vertically movable, so that when its downward displacement is y the extension of the spring is $(x - y)$. The equation of motion is

$$m\ddot{x} = mg - k(x - y) - p\dot{x}$$

or,

$$m\ddot{x} + p\dot{x} + kx = mg + ky.$$

If we suppose the upper end of the spring has a forced periodic motion, we can put $y = a \sin \omega t$, having an amplitude a and a frequency $\omega/2\pi$. The equation becomes

$$(mD^2 + pD + k)x = mg + ka \sin \omega t.$$

If the damping p is not too great, the complementary function gives damped free vibrations. Part of the particular integral is $x = mg/k$, giving the extension in the equilibrium position. The other part is

$$x = \frac{ka}{mD^2 + pD + k} \sin \omega t$$

$$= \frac{ka}{(k - m\omega^2) + pD} \sin \omega t$$

$$= \frac{ka \sin (\omega t - \phi)}{\sqrt{\{(k - m\omega^2)^2 + p^2\omega^2\}}}, \qquad \tan \phi = \frac{p\omega}{k - m\omega^2}.$$

The rest of the discussion follows lines parallel to the previous electrical part.

EXERCISES

1. A series circuit with an alternating voltage has $R = 200$ ohms, $L = 500$ mH, $C = 10\ \mu$F. By solving the equation for the current, estimate the frequency which gives a lead of 45° over the voltage. [About 46 cycles per sec.]

2. A 50 gm. mass hangs from a vertical spring of stiffness 20 gm. wt./cm. The damping is 1·4 gm. wt. per cm./sec. velocity. The support oscillates vertically with frequency 10 per second and amplitude 1 cm. Find the amplitude of the forced oscillations of the mass.

3. A rotor with inertia I is driven by a couple which varies with the time, and may be taken as $A \sin \omega t$. The frictional couple in the bearings may be taken as proportional to the angular velocity and of magnitude $P\dot{\theta}$. Prove that ultimately the rotor makes forced oscillations of frequency $\omega/2\pi$ per second, and determine their amplitude. $\left[\dfrac{A}{\omega\sqrt{(P^2 + I^2\omega^2)}}.\right]$

4. A simple pendulum of length λ carries a bob of mass m. If the support makes forced lateral vibrations of amplitude c and frequency $\omega/2\pi$, discuss the motion of the bob. Consider in particular the resonance case.

$$\left[-mg\theta = \frac{d^2}{dt^2}(\lambda\theta + c\sin\omega t).\right]$$

5. If in the series circuit, discussed in the text above, the applied voltage is damped and has the form $E_0 e^{-\beta t}\sin\omega t$, prove that if the resistance is not too great the voltage consists of two damped vibrations superposed. The problem is of interest in the theory of wireless telgraphy.

6. If the β, ω of the last problem are respectively the b, c of the text, the equation for the voltage is

$$(D^2 + 2bD + n^2)V = Ae^{-bt}\sin ct.$$

Prove that in this case the amplitude is proportional to te^{-bt}. If b is very small, this may become temporarily large, even though it finally becomes zero.

APPENDIX TO CHAPTER III

The reduced linear equation of the second order cannot have more than two linearly independent solutions. (See 3, 1.)

The proposition is established by showing that the assumption of three linearly independent solutions leads to a contradiction. The reader will recall that the two simple simultaneous equations

$$ax + by + c = 0, \quad px + qy + r = 0,$$

usually furnish unique values for x and y. There are two exceptions to this. In the first case, if

$$\frac{a}{p} = \frac{b}{q} \neq \frac{c}{r},$$

the equations are inconsistent, and no values of x and y are capable of satisfying the equations simultaneously. In the second case, if

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r},$$

the equations are consistent, but they are effectively one and the same equation. The value of x can be arbitrarily assigned, and there is a unique corresponding value of y.

Let y_1, y_2 and y_3 be solutions of the reduced equation

$$(1) \quad ay + by' + cy'' = 0.$$

If constants l, m, n exist (not all zero) such that

$$(2) \quad ly_1 + my_2 + ny_3 \equiv 0,$$

the solutions are said to be linearly related. If no such constants exist, the solutions are said to be linearly independent. The contention is that any three such solutions are necessarily linearly related.

Assuming that they are independent, we have, by hypothesis,

$$(3) \quad ay_1 + by_1' + cy_1'' = 0,$$

$$(4) \quad ay_2 + by_2' + cy_2'' = 0,$$

$$(5) \quad ay_3 + by_3' + cy_3'' = 0.$$

If we multiply the first by λ, the second by μ, and add, it must be possible to choose λ and μ so that the coefficients of a, b are zero, or

$$(6) \quad \lambda y_1 + \mu y_2 + y_3 = 0,$$

$$(7) \quad \lambda y_1' + \mu y_2' + y_3' = 0.$$

To deny this is to assert that

$$\frac{y_1'}{y_1} = \frac{y_2'}{y_2}.$$

This last equation is equivalent to $\log y_1 = \log y_2 + \log k$, or $y_1 = ky_2$. Having reached a result that violates our assumption of independence, we are driven to the conclusion that λ and μ are determinable; but notice that so far there is no reason for supposing them to be constants.

The differentiation of (6) gives

$$\lambda y_1' + \lambda' y_1 + \mu y_2' + \mu' y_2 + y_3' = 0,$$

whence by subtraction of (7) we have

$$(8) \quad \lambda' y_1 + \mu' y_2 = 0.$$

Meantime the coefficient of c is automatically zero, and we have the additional equation

$$\lambda y_1'' + \mu y_2'' + y_3'' = 0.$$

Taken in conjunction with (7) this gives similarly

$$(9) \quad \lambda' y_1' + \mu' y_2' = 0.$$

Concerning (8) and (9), the possibility

$$\frac{y_1'}{y_1} = \frac{y_2'}{y_2}$$

is ruled out. We are accordingly left with

$$\lambda' = 0 = \mu',$$

so that λ, μ are absolute constants and in accordance with (6) the three solutions are in fact connected by a linear relation. It only remains to add that as no assumptions were made regarding a, b and c, the proof is valid whether they be constants or functions of x.

CHAPTER IV

Miscellaneous Theorems and Methods

4, 1. *Orthogonal Trajectories.*

The present chapter consists mainly of a number of unrelated theorems which are not covered by the previous analysis and have to be fitted in somewhere. Their applicability is limited in practice, but students are usually expected to show a knowledge of them. We begin with what are known as orthogonal trajectories.

A family of curves that depends on a single parameter is said to be singly infinite in number. If two singly infinite families of curves are such that every member of the one family crosses every member of the other family at right angles, the relation is reciprocal, and the families are said to constitute orthogonal trajectories. A simple example is afforded by a sheet of squared paper, where the vertical family $x = a$ is orthogonal to the horizontal family $y = b$. Similarly, the set of radial lines $y = mx$ through the origin is orthogonal to the concentric circles $x^2 + y^2 = r^2$; whilst in a plane field of force the lines of force are orthogonal to the equipotential lines. It is possible to have non-orthogonal trajectories, where the constant angle of intersection is other than $\frac{1}{2}\pi$; and the idea can be extended to families on a curved surface, e.g. lines of longitude and parallels of latitude.

Considering the two curves, one from each family, that pass through any point, the slope of the one curve is variously denoted by

$$\frac{dy}{dx} = \tan\psi = m.$$

If the slope of the other curve be denoted by m', the known condition for perpendicularity is $mm' = -1$. This means that the dy/dx for the one curve is the negative reciprocal of the dy/dx for the other curve. The procedure for finding the orthogonal trajectories of a given one-parameter family now becomes clear. We first eliminate the parameter and so derive the differential equation of the family. On replacing dy/dx by its negative reciprocal we achieve the differential equation of the other family. Integration then leads to the cartesian form.

If we happen to be working in polar co-ordinates, the angle ϕ between a curve and its radius vector is given by

$$\tan\phi = r\frac{d\theta}{dr}.$$

If the same radius vector makes angle ϕ' with the corresponding curve from the other family, the relation between ϕ and ϕ' is

$$\phi' = \phi + \tfrac{1}{2}\pi, \quad \tan\phi\tan\phi' = -1.$$

The corresponding procedure is therefore to replace

$$r\frac{d\theta}{dr} \quad \text{by} \quad -\frac{1}{r}\frac{dr}{d\theta},$$

or, what comes to the same thing, replace

$$\frac{d\theta}{dr} \quad \text{by} \quad -\frac{1}{r^2}\frac{dr}{d\theta}.$$

Example 1.—The radial lines $y = mx$ give $dy/dx = m$, and hence

$$y = x\frac{dy}{dx}$$

is the differential equation of the family. The orthogonal family would accordingly be

$$y = -x\frac{dx}{dy}, \quad x\,dx + y\,dy = 0.$$

The solution of this is $$x^2 + y^2 = r^2,$$
a family of concentric circles.

Example 2.—Consider the circles defined by $r = a\cos\theta$ which all pass through the origin, and have their centres on the initial line, a being the variable diameter. Logarithmic differentiation gives

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan\theta$$

as the differential equation of the family. The orthogonal family would be defined by

$$r\frac{d\theta}{dr} = \tan\theta, \quad \frac{dr}{r} = \cot\theta\,d\theta.$$

The variables are separated and integration gives

$$\log r = \log\sin\theta + \log c,$$

or $r = c\sin\theta$. This is another family of circles through the origin and touching the initial line.

In forming the differential equation of the orthogonal family by replacing (dy/dx) by $-(dx/dy)$, it is not inconceivable that the differential equation may be unchanged. Such would be the case, for example, with the equation

$$p^2(x + y) = (1 + p^2)^2, \quad p = \frac{dy}{dx}.$$

The modified differential equation could then integrate back only to the original from which it derived. This means that the family is its own orthogonal trajectory, or is self-orthogonal. Failure to recognize the possibility of this usually arises through taking too narrow a view of a family. The equation

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$$

is frequently regarded as a family of confocal ellipses; but if we allow λ to take all values, positive or negative, the family includes the corresponding hyperbolas as well as both co-ordinate axes.

Example 3.—It is known from a previous exercise (see I, 8, No. 5) that the above family of confocal conics, where λ is the variable parameter, has the differential equation

$$p(a^2 - b^2) = (x + yp)(xp - y).$$

It is readily verified that the substitution of $-1/p$ for p makes no change. We avoid the trouble of integration by saying that the equation must integrate back to the original family. A rough sketch of the ellipses and hyperbolas will indicate the nature of the problem.

If we write $z = x + iy$, where x, y are independent variables, it is well known that any function of z necessarily falls into two parts, of which one is real and the other purely imaginary. It is customary to write

$$f(z) = w = u + iv,$$

where, of course, u and v are functions of x and y. As simple illustrations we have

$$(1) \quad w = z^2, \quad u = x^2 - y^2, \quad v = 2xy.$$

$$(2) \quad w = \log z, \quad u = \tfrac{1}{2}\log(x^2 + y^2), \quad v = \tan^{-1}\frac{y}{x}.$$

The problem of separating such functions into their real and imaginary parts is usually treated in the higher parts of texts on trigonometry. The families defined by giving constant values to u and v are orthogonal, a property which makes them of outstanding importance in such subjects as electrostatics and hydrodynamics. Taking the second illustration above, the constancy of u implies the constancy of x^2+y^2, which is a circle; and the constancy of v gives the constancy of y/x, which is a straight line through the origin. We revert, in fact, to the Example 1 already worked out.

In view of the independence of x and y, we have in general

$$\frac{\partial z}{\partial x}=1, \quad \frac{\partial z}{\partial y}=i,$$

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial w}{\partial x}=\frac{\partial w}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial w}{\partial z},$$

$$\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}=\frac{\partial w}{\partial y}=\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial y}=i\frac{\partial w}{\partial z}.$$

The last two lines yield

$$\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}=i\left\{\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right\}.$$

Equating the real parts, and also the imaginary parts, we reach

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

These last two equations are known as *the Riemann-Cauchy relations*.

When we rob x and y of part of their independence and compel them to run in harness by equating $u(x, y)$ to a parametric constant, the slope is given by differentiating an implicit function, and we have

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0,$$

so that
$$\left(\frac{dy}{dx}\right)_u=-\left(\frac{\partial u}{\partial x}\right)\Big/\left(\frac{\partial u}{\partial y}\right).$$

Similarly,
$$\left(\frac{dy}{dx}\right)_v=-\left(\frac{\partial v}{\partial x}\right)\Big/\left(\frac{\partial v}{\partial y}\right).$$

In virtue of the Riemann-Cauchy relations, the product of these two slopes is evidently -1, so that the two families are at right angles. Hence any number of orthogonal families can be obtained by equating to parametric constants the real and imaginary parts of functions of the complex variable $(x+iy)$.

EXERCISES

1. Find the orthogonal trajectories of the Boyle's law curves $PV =$ constant. [$P^2 - V^2 =$ const.]

2. A family of circles, with centres on OX, make intercepts $\pm c$ on OY, so that all the circles pass through the same two points. Prove their orthogonal trajectories to be a family of circles, with centres on OY, having these two points as real limiting points.

3. The family of circles $x^2 + y^2 = 2cy$ touches OX at the origin. The orthogonal family is circles touching OY at the origin.

4. Consider the family of parabolas with focus at the origin and vertex on OX. Find their cartesian equation and hence their differential equation. How do you interpret the result of trying to find their orthogonal trajectories?

5. $r = a \sin^2\theta$ leads to $r^2 = b \cos\theta$.

6. $r = a \operatorname{cosec}^2 \frac{1}{2}\theta$ leads to $r = b \sec^2 \frac{1}{2}\theta$.

7. Find the orthogonal trajectories of $r = a\theta$.

8. Prove that the family of semi-cubical parabolas $cy^2 = x^3$ gives a family of similar and similarly situated ellipses.

9. Apply the principles of the text by discussing the families defined by the transformation $w = \cos z$.

4, 2. *One Variable Absent.*

It occasionally happens that one of the variables does not appear explicitly in the equation. It may even happen that both variables are absent. In these cases the equation can be somewhat simplified by lowering its order. We begin with the case where the dependent variable is absent.

When y is absent the equation has the form

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right) = 0.$$

We choose a new dependent variable by writing $dy/dx = p$, so that $y'' = p'$, &c. The equation then takes the form

$$F(x, p, p', \ldots) = 0,$$

and the order is now lower by unity. Theoretically, we get a solution either in the form $p = \phi(x)$ or $x = \psi(p)$. In the former case we replace p by dy/dx, so that $dy/dx = \phi(x)$, and y is given explicitly as a function of x. In the second case, differentiation with respect to x gives

$$1 = \psi'(p) \frac{dp}{dx},$$

but as
$$\frac{dp}{dx} = p\frac{dp}{dy}$$
we have
$$y = \int p\psi'(p)\,dp.$$

The elimination of p would give a relation between x and y; but in practice it is just as convenient to omit the elimination and regard the results as giving x and y parametrically in terms of p.

Example 1.—$\frac{dy}{dx}\cos x = \frac{d^2y}{dx^2}\sin x.$

This can be written in the form
$$\cot x = \frac{1}{p}\frac{dp}{dx}.$$
Hence on integration
$$\log\sin x + \log c = \log p$$
or
$$p = c\sin x = \frac{dy}{dx}.$$
The solution is therefore
$$y = a - c\cos x.$$

Example 2.—$\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = e^x.$

Here x is immediately given parametrically as
$$x = \log(p + p^2).$$
Differentiation gives
$$1 = \frac{1+2p}{p+p^2}\frac{dp}{dx} = \frac{1+2p}{1+p}\frac{dp}{dy},$$
leading to
$$dy = \left(2 - \frac{1}{1+p}\right)dp,$$
whence on integration
$$y = 2p - \log(1+p) + c.$$
The result is left in parametric form. There is little hope of eliminating p.

When the independent variable x is absent, the equation takes the form
$$F\left(y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \ldots\right) = 0.$$
As before, we put $dy/dx = p$; but for the next step we write
$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = p\frac{dp}{dy},$$
and so on for higher orders, if need be. The equation takes the form
$$F\left(y,\ p,\ p\frac{dp}{dy},\ \ldots\right) = 0,$$

and the order is lowered by unity. We reach a solution either in the form $p = \phi(y)$ or $y = \psi(p)$. In the former case we replace p by dy/dx, and the variables are separated. In the alternative case we differentiate with respect to x, and have

$$p = \frac{d\psi}{dp}\frac{dp}{dx},$$

so that the variables are again separable.

Example 3.—Consider the non-linear equation $y\dfrac{d^2y}{dx^2} = 1 + \left(\dfrac{dy}{dx}\right)^2$.

The procedure gives $$yp\frac{dp}{dy} = 1 + p^2,$$

and on separating the variables we reach

$$\log(1 + p^2) = \log y^2 - \log c^2,$$

whence $$cp = \sqrt{(y^2 - c^2)} = c\frac{dy}{dx}.$$

Separating the variables again we have

$$dx = \frac{c\,dy}{\sqrt{(y^2 - c^2)}},$$

whence $$x + a = c\cosh^{-1}\frac{y}{c}, \quad y = c\cosh\frac{x + a}{c},$$

the catenary.

In certain commonly occurring cases the given equation has such a form that other methods are readily applicable. Thus the motion of a body acted on by a force varying with the time might have the equation

$$m\frac{d^2x}{dt^2} = a\sin(\omega t + \phi),$$

where x is absent. The solution would be obtained by direct integration. If the force depended on the distance rather than the time, we might have

$$m\frac{d^2x}{dt^2} = a\sin(nx + \phi)$$

with t absent. Two procedures are now available. We can multiply both sides by dx/dt and integrate. This gives

$$\tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 = c - \frac{a}{n}\cos(nx + \phi),$$

which is the equation of energy. Alternatively, we can write the acceleration in the form

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx}.$$

The transformed equation

$$mv\frac{dv}{dx} = a\sin(nx + \phi)$$

has the variables separable and we again reach the equation of energy.

It must be borne in mind that all we can be sure of when one variable is absent, or both, is that the order of the equation can be reduced by unity. There is no guarantee that even a first integral is then obtainable; still less is it certain that the equation is soluble.

EXERCISES

1. $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$ $[y = a + b\sinh^{-1}x.]$

2. $y\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2.$ $\left[y = \frac{c}{x + a}.\right]$

3. Solve the following equations from the theory of potential:

(i) $\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$ $\left[V = b - \frac{a}{r}.\right]$

(ii) $\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0.$ $[V = a + b\log r.]$

4. The radius of curvature of a curve is equal to the normal. If they are on opposite sides, prove that the curve is a catenary. If they are on the same side, verify the geometrically obvious answer that the curve is a circle with its centre on OX.

5. Solve in parametric form

$$\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = e^y. \quad \left[x = 2\tan^{-1}p - \frac{1}{p} + c.\right]$$

6. Write down the differential equation of all curves whose radius of curvature is constant and solve it.

7. Find a curve with a constant normal.

8. Solve the equation $\frac{d^2x}{dt^2} = a - n^2x$ and interpret the result.

9. The equation

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$$

occurs in the theory of orbits. The interpretation is that u is the inverse of the distance r, h is the constant angular momentum, and P is the central force which obeys some law of distance. Solve the equation when P is equal to (i) μu^2; (ii) μu^3. The matter is treated in any text dealing with particle dynamics.

10. A particle of mass m moves in a straight line under an attraction to a fixed point on the line, the force being inversely proportional to the square of the distance. The equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}.$$

If the particle is moving away from the centre of attraction with velocity u when at distance h, what decides whether it will ever return?

11. The tangent PT and the normal PN at any point P of a curve meet OX in T, N respectively. Find parametrically the curve for which the length of TN is constant.

12. All rectangular hyperbolas with their asymptotes parallel to the coordinate axes (see 1, 8, No. 10) satisfy the equation

$$2\frac{dy}{dx}\frac{d^3y}{dx^3} = 3\left(\frac{d^2y}{dx^2}\right)^2.$$

Integrate this equation.

4, 3. *First Order Equations of Degree Higher than the First.*

A first order equation may have the form

$$P_0 + P_1p + P_2p^2 + \ldots + P_np^n = 0,$$

where P_0, P_1, &c., are functions of x and y. If specific values be given to x and y the coefficients take numerical values, and we have an equation of degree n to find p, which in general has n distinct values. The interpretation is that in general n branches pass through any point. Some of these may be imaginary; or there may be a real locus along which two or more values of p are equal, a matter of interest in the higher theory.

As a rule, little can be done with this form unless it can in some way be factorized as a preliminary step. If we can write the equation as

$$(p-f_1)(p-f_2)\ldots(p-f_n) = 0,$$

where f_1, f_2, &c., are functions of x and y, the answer to the problem is the aggregate of solutions of the separate equations

$$\frac{dy}{dx} = f_1 \text{ or } f_2 \text{ or } f_3, \text{ etc.}$$

Example.—$p^2 - px + py - xy = 0$.

This is equivalent to $$(p - x)(p + y) = 0,$$

so that $$\frac{dy}{dx} = x \quad \text{or} \quad \frac{dy}{dx} = -y.$$

The solution is accordingly the pair of results
$$y = \tfrac{1}{2}x^2 + c, \quad y = ce^{-x}.$$

EXERCISES

1. $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}.$ $[xy = c, \quad x^2 - y^2 = c.]$

2. $p^2 + 2p \sinh x = 1.$ $[y = c - e^x, \quad y = c - e^{-x}.]$

3. Prove that if the equation can be solved as $y = f(x, p)$, differentiation with respect to x will give an equation in x and p, and thus give x and y parametrically in terms of p. Hence solve the equation
$$\frac{dy}{dx} = \sqrt{(2y - x + 1)}.$$
$[x = p + \tfrac{1}{2}\log(2p - 1) + c.]$

4. Prove that a similar procedure will apply if the equation can be solved as $x = f(y, p)$, and we differentiate with respect to y. Apply the method to the equation
$$x\left(\frac{dy}{dx}\right)^2 = a + \frac{dy}{dx}. \qquad \left[y = c + \frac{2a}{p} - \log p.\right]$$

4, 4. *Clairaut's Form.*

An outstanding instance of the theorem mentioned in Ex. 3 above is the equation
$$y = px + f(p).$$
It is known as Clairaut's form. If we remember that $p = \tan\psi$ and that $y - px$ is the intercept on OY, the equation gives a relation between the intercept and the slope. The relation must accordingly hold throughout the length of any line whose intercept and slope fit the given relation. But along any such line, p is constant, so that the line would be
$$y = cx + f(c).$$
Treating the problem by the suggested method of differentiation with respect to x, we have
$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}.$$

This entails $dp/dx = 0$, in which case we have $p = c$ and $y = cx + f(c)$ as before. Alternatively,

$$x + f'(p) = 0,$$

and the elimination of p between this and the original equation gives a specific relation between x and y. It will be noticed that the result must be the same as that obtained by eliminating c between

$$y = cx + f(c) \quad \text{and} \quad 0 = x + f'(c);$$

the point of which is that, according to texts on the calculus, this is the method of finding the envelope of the family of straight lines. Since the envelope is not one of the family and cannot be derived by giving a specific value to c, it is known as a "singular solution". It involves no arbitrary constants and the equations

$$x = -f'(p), \quad y = px + f(p)$$

really give the co-ordinates parametrically in terms of p or ψ.

Example.—$y = px + \dfrac{a}{p}$.

This has the solution $\qquad y = cx + \dfrac{a}{c}$.

The envelope is obtained by eliminating p with the aid of

$$0 = x - \frac{a}{p^2} \quad \text{or} \quad 0 = px - \frac{a}{p}.$$

On subtracting the squares of both equations in p, we have $y^2 = 4ax$. Thus the ordinary solution is a straight line, whilst the singular solution is its parabolic envelope. The whole problem corresponds to the fact that any point on the parabola $y^2 = 4ax$ can be expressed parametrically as $x = at^2$, $y = 2at$.

EXERCISES

1. Find the singular solution of

$$y = px - \frac{ap^2}{1 + p}. \qquad [(x + y)^2 = 4ay.]$$

2. Prove that the singular solution of

$$y = px + \sqrt{(b^2 + a^2p^2)}$$

is an ellipse.

3. The tangent to a curve makes a triangle of constant area with the co-ordinate axes. Prove that the curve is a rectangular hyperbola.

4. If the tangent to a curve meets the co-ordinate axes in A, B, prove that the length of AB is $(px - y)\sqrt{(1 + p^2)}/p$. Deduce the form of the curve for which this is constant.

4, 5. *The Homogeneous Linear Equation.*

A form that usually receives notice, though of limited applicability, is the equation

$$ay + bx\frac{dy}{dx} + cx^2\frac{d^2y}{dx^2} + \ldots + hx^n\frac{d^ny}{dx^n} = f(x). \quad . \quad . \quad \text{(i)}$$

It is sometimes called a homogeneous equation, though foreign texts use the word in a different sense. The reason for the name becomes apparent if we make the substitution $y = Ax^m$. Every term in the auxiliary equation then becomes a multiple of x^m. This suggests a method of finding the complementary function, for our substitution will be valid if

$$a + bm + cm(m-1) + \ldots = 0.$$

This in general gives n suitable values of m. Moreover, if $f(x)$ has the form cx^r, we might hope to find a particular integral by the substitution $y = Ax^r$. The result of this substitution is

$$\{a + br + cr(r-1) + \ldots\}Ax^r = cx^r,$$

which determines A.

Example 1.—$2y - 2x\dfrac{dy}{dx} + x^2\dfrac{d^2y}{dx^2} = x^3.$

For the complementary function we substitute $y = x^m$, which gives

$$2 - 2m + m(m-1) = 0 = m^2 - 3m + 2,$$

so that $m = 1, 2$. The complementary function is

$$y = ax + bx^2.$$

For the particular integral we substitute $y = Ax^3$. This gives

$$(2 - 2 \cdot 3 + 3 \cdot 2)Ax^3 = x^3,$$

whence $A = \frac{1}{2}$. The full solution is therefore

$$y = ax + bx^2 + \tfrac{1}{2}x^3.$$

A little reflection shows that the above method, appealing in its simplicity, bristles with objections. What happens if the equation for m has coincident or complex roots; or if r is less than n? The necessity for conducting a discussion on these points can be obviated by a simple procedure, one that replaces the equation by another with constant coefficients and thus places at our disposal our previously acquired knowledge of such equations. We make the substitution

$$x = e^{\theta}, \quad \frac{dx}{d\theta} = x.$$

We then have
$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = x\frac{dy}{dx}.$$

If, as is customary, we denote $d/d\theta$ by ϑ and continue to use D for d/dx, the last relation shows that ϑ is equivalent to xD. It remains to examine the form of repeated operations; and here we might as well point out the common error of supposing that $(xD)^2$ is x^2D^2. We accordingly consider the identity

$$D(x^rD^r) \equiv x^rD^{r+1} + rx^{r-1}D^r.$$

On multiplying by x we have

$$xD(x^rD^r) = x^{r+1}D^{r+1} + rx^rD^r,$$

which can be arranged as

$$x^{r+1}D^{r+1} = (xD - r)x^rD^r.$$

On putting $r = 1$, we have

$$x^2D^2 = (xD - 1)xD = (\vartheta - 1)\vartheta.$$

Similarly $r = 2$ gives

$$x^3D^3 = (xD - 2)x^2D^2 = (\vartheta - 2)(\vartheta - 1)\vartheta,$$

and so on. The original equation (i) now becomes

$$\{a + b\vartheta + c\vartheta(\vartheta - 1) + \ldots\}y = f(e^{\theta})$$

or
$$F\left(\frac{d}{d\theta}\right)y = f(e^{\theta}),$$

a linear equation with constant coefficients.

Example 2.—Taking as before $2y - 2x\dfrac{dy}{dx} + x^2\dfrac{d^2y}{dx^2} = x^3$,

we have
$$\{2 - 2\vartheta + \vartheta(\vartheta - 1)\}\, y = e^{3\theta} = \{\vartheta^2 - 3\vartheta + 2\}y.$$

The symbol ϑ now plays the role usually allotted to D. The roots for ϑ are 1, 2. The complementary function is

$$y = Ae^{\theta} + Be^{2\theta}$$

and the particular integral is

$$y = \frac{1}{\vartheta^2 - 3\vartheta + 2}\, e^{3\theta} = \tfrac{1}{2}e^{3\theta}.$$

The solution is
$$y = Ae^{\theta} + Be^{2\theta} + \tfrac{1}{2}e^{3\theta}.$$
We now replace x by the relations
$$x = e^{\theta}, \quad \theta = \log x,$$
and reach
$$y = Ax + Bx^2 + \tfrac{1}{2}x^3$$
as before.

Example 3.—$x^3 \dfrac{d^2y}{dx^2} + 3x^2 \dfrac{dy}{dx} + xy = 1.$

We re-write as
$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{x}.$$
In terms of θ and ϑ this becomes
$$\{\vartheta(\vartheta - 1) + 3\vartheta + 1\}y = e^{-\theta} = (\vartheta + 1)^2 y.$$
For the complementary function there is a repeated root $\vartheta = -1, -1$, and we have
$$y = (A + B\theta)e^{-\theta}.$$
The particular integral is given symbolically as
$$y = \frac{1}{(\vartheta + 1)^2}\, e^{-\theta} = e^{-\theta}\frac{1}{\vartheta^2}1 = \tfrac{1}{2}\theta^2 e^{-\theta}.$$
The solution is
$$y = (A + B\theta)e^{-\theta} + \tfrac{1}{2}\theta^2 e^{-\theta}.$$
In terms of x this becomes
$$xy = A + B\log x + \tfrac{1}{2}(\log x)^2.$$

EXERCISES

1. Solve $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 6x^5.$ $[y = Ax^2 + Bx^3 + x^5.]$

2. The equation
$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$
occurs in the theory of thick cylinders. Prove by both methods that its solution is
$$u = Ar + \frac{B}{r},$$
and that
$$u = r\frac{d}{dr}\left(r\frac{du}{dr}\right)$$
is another form of the same equation.

3. The depression of a thick circular disc under normal pressure is approximately governed by the equation
$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = -Nr.$$

Prove that it has the solution

$$u = Ar + \frac{B}{r} - \frac{Nr^3}{8}.$$

4. The depression of a thick circular disc with central load P is governed by

$$\frac{d^2\varphi}{dx^2} + \frac{1}{x}\frac{d\varphi}{dx} - \frac{\varphi}{x^2} = -\frac{P}{x}.$$

Derive the solution $$\varphi = Ax + \frac{B}{x} - \tfrac{1}{2}Px\log x.$$

5. In investigations on the strength of rotating cylinders we meet the equation

$$u = r\frac{d}{dr}\left(r\frac{du}{dr}\right) + Hr^3.$$

Prove that $$u = Ar + \frac{B}{r} - \frac{Hr^3}{8}.$$

6. Solve the equations (i) $\dfrac{d^2V}{dr^2} + \dfrac{1}{r}\dfrac{dV}{dr} = 0,$

(ii) $\dfrac{d^2V}{dr^2} + \dfrac{2}{r}\dfrac{dV}{dr} = 0,$

which occur in the theory of potential.

7. Solve the equation $$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 4\pi\rho$$

which occurs in the discussion of the space-charge law for electron tubes.

CHAPTER V

Simultaneous Equations

5, 1. Hitherto we have discussed only equations in which there was one variable dependent on another. We now turn to sets of equations in which there is one independent variable, usually taken as the time, and two or more other variables dependent on it. The number of given equations must equal the number of dependent variables; and the problem is to express each dependent variable as an explicit function of the time, subject to any prescribed conditions, so that each variable is computable for a given instant.

In the simplest case each equation may contain just two variables. They are then separately soluble.

Example 1.—A mass m moves in a plane under an attraction to the origin proportional to the distance and of magnitude nr. The arrangement could be realized in practice by an elastic string. The components of this force parallel to the co-ordinate axes are nx and ny. The equations of motion parallel to the axes are

$$m\ddot{x} = -nx, \quad m\ddot{y} = -ny.$$

The solutions are

$$x = A\cos\omega t + B\sin\omega t, \quad m\omega^2 = n,$$
$$y = C\cos\omega t + D\sin\omega t.$$

Regarded as a problem in simultaneous equations, that is the end of the matter. If we care to solve the equations we get $\cos\omega t$ and $\sin\omega t$ expressed as linear functions of x and y; hence on squaring and adding we get a quadratic in x and y equal to unity. This must be a conic; but as x is certainly not numerically greater than $(A + B)$, nor y than $(C + D)$, the mass makes only a limited excursion from the origin and the conic must be an ellipse. Under special conditions this might be either a circle or a straight line. The investigation of these conditions is left to the reader. It may be as well to point out that, had the path of the mass been the primary object of our investigation, other methods would have been available.

If we eliminate n from the differential equations, we get

$$m(x\ddot{y} - y\ddot{x}) = 0 = m\frac{d}{dt}(x\dot{y} - y\dot{x}),$$

which proves that the angular momentum $m(x\dot{y} - y\dot{x})$ is constant. Alternatively, if we multiply the differential equations respectively by $\dot{x}$, $\dot{y}$, and add, we have

$$m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + n(x\dot{x} + y\dot{y}) = 0,$$

which on integration gives

$$\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}n(x^2 + y^2) = c.$$

This is the equation of energy.

In certain other cases the treatment is almost equally simple.

Example 2. In an investigation of the motion of an engine governor partially controlled by a spring, equations of the following type occur:

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = kx,$$

$$\frac{dx}{dt} + cy = d.$$

The obvious procedure is to differentiate the first equation and replace dx/dt from the second. This gives

$$\frac{d^3y}{dt^3} + a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + kcy = kd.$$

When once y is determined from this equation, the first equation determines x uniquely.

In the treatment of such cases as usually occur we employ algebraic manipulation, possibly in conjunction with differentiation, to eliminate all the dependent variables except one. We are then left with an ordinary equation in two variables. In the case of the other variables it is not generally necessary, nor even desirable, to repeat the process. They are better found by retracing the steps of the elimination. An example will illuminate the matter.

Example 3.—$\dfrac{dx}{dt} + 8x + y = 0,$

$$\frac{dy}{dt} - 4x + 3y = 0.$$

Suppose we decide to eliminate y. As only three equations are needed to eliminate the two unknowns y and dy/dt, a single differentiation should suffice. We accordingly have

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + \frac{dy}{dt} = 0$$

from the first equation. The next step is pure algebra, and leads to

$$\frac{d^2x}{dt^2} + 11\frac{dx}{dt} + 28x = 0.$$

The roots are $D = -4, -7$, and we have $x = Ae^{-4t} + Be^{-7t}$. It is left to the reader to verify that the elimination of x would give the exactly similar equation

$$\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 28y = 0,$$

so that $y = Ce^{-4t} + De^{-7t}$.

We are now apparently in possession of four arbitrary constants; but if we choose to substitute back in the original equations, we get two relations between the constants and can thus determine any pair in terms of the other pair, so that only two are really independent. This labour can be avoided by a judicious use of the equations. From the first of them we have

$$y = -\frac{dx}{dt} - 8x = -4Ae^{-4t} - Be^{-7t}.$$

This gives y explicitly and the number of arbitrary constants is only two.

The matter is more expeditious if we use the symbolic method. We then write

$$(D+8)x + y = 0, \quad 4x - (D+3)y = 0.$$

To eliminate y we "multiply" the first equation by $(D+3)$ and add. This gives

$$(D+3)(D+8)x + 4x = 0 = (D^2 + 11D + 28)x,$$

whence $D = -4, -7$, and the rest proceeds as before.

In the last example there might very well have been terms in t on the right. This makes no difference to the procedure, and merely introduces a particular integral into the solution.

Example 4.—$$\frac{dx}{dt} + 8x + y = 20t^2 - 60t - 187,$$

$$\frac{dy}{dt} - 4x + 3y = -24t^2 + 102t - 30.$$

By operating on the first equation with $\left(\frac{d}{dt} + 3\right)$, we have

$$\frac{d^2x}{dt^2} + 11\frac{dx}{dt} + \frac{dy}{dt} + 24x + 3y = 60t^2 - 140t - 621.$$

Subtracting the second equation, we have

$$\frac{d^2x}{dt^2} + 11\frac{dx}{dt} + 28x = 84t^2 - 242t - 591.$$

According to 3, 8, No. 10, this has the particular integral $x = 3t^2 - 11t - 17$, so that

$$x = Ae^{-4t} + Be^{-7t} + 3t^2 - 11t - 17,$$

whence

$$y = -4Ae^{-4t} - Be^{-7t} - 4t^2 + 22t - 40.$$

Unless the equations in general have the relatively simple forms exemplified above, the difficulties of elimination may become insuperable. Fortunately most of the commoner equations are of this simple type, and we shall now study a few cases in detail.

Consider the case of two heavy rotors on the same shaft, such as might be a turbine and a dynamo. Let the rotor on the left have inertia I, with J for the other rotor. When the system is stationary we mark on each rotor a vertically upward radius to serve as an indicator. Suppose that in any subsequent motion these radii are respectively displaced through angles θ, ϕ to the right of the vertical, as viewed from the right hand, so that the motion is clockwise. As a matter of convenience we will assume ϕ greater than θ. The angular twist on the shaft is $(\phi - \theta)$, and if the rigidity is such that C is the couple required to give unit twist, the couple is $C(\phi - \theta)$. Its effect tends to increase θ and diminish ϕ.

If we suppose that the system is running under no effective applied couple, which means that there is more or less steady running with the power paired-off with the resistance, the separate equations of motion are

$$I\ddot{\theta} = C(\phi - \theta),$$

$$J\ddot{\phi} = -C(\phi - \theta).$$

The first thing we notice is that by addition

$$I\ddot{\theta} + J\ddot{\phi} = 0.$$

This means that the rate of change of angular momentum is zero; or the angular momentum is constant. This accords with our assumption that there is no effective applied couple. If we denote the constant angular momentum by $(I + J)\omega$, so that

$$I\dot{\theta} + J\dot{\phi} = (I + J)\omega,$$

then ω is the mean angular speed of either rotor or of the whole system. It will appear later that each rotor fluctuates about this mean speed.

We now write the equations of motion in the symbolic form,

$$(ID^2 + C)\theta - C\phi = 0,$$

$$C\theta - (JD^2 + C)\phi = 0.$$

It is immaterial whether we eliminate θ or ϕ; the equation is the same in either case. Presuming that we eliminate ϕ, we have

$$\left\{(ID^2 + C)(JD^2 + C) - C^2\right\}\theta = 0 = \left\{IJD^2 + C(I + J)\right\}D^2\theta.$$

The four roots for D are two zeros and a pair of conjugate imaginaries; so that, if for brevity we put $C(I + J)/IJ = \beta^2$, we have $D = 0, 0, \pm i\beta$, and

$$\theta = A + Bt + P\cos\beta t + Q\sin\beta t.$$

The constant A is of no consequence since it can be varied at will by measuring θ from some initial line other than the vertical. The angular velocity is

$$\dot{\theta} = B + \beta(Q \cos \beta t - P \sin \beta t).$$

If we consider the mean value of this over a long period of time, the trigonometrical terms (being alternately positive and negative) have a mean value of zero. The interpretation of B is that it is the mean angular velocity, previously denoted by ω. To anticipate slightly, the equation really gives the Fourier expansion of the angular velocity.

Hence $$\theta = A + \omega t + P \cos \beta t + Q \sin \beta t.$$

To determine ϕ we substitute in the first equation of motion, whence

$$\begin{aligned}\phi &= \theta + I\ddot{\theta}/C \\ &= A + \omega t + \left(1 - \frac{I\beta^2}{C}\right)\{P \cos \beta t + Q \sin \beta t\} \\ &= A + \omega t - \frac{I}{J}\{\mathrm{P} \cos \beta t + Q \sin \beta t\}.\end{aligned}$$

The constant A can be removed from both equations by advancing the initial line through angle A from the vertical. Evidently both rotors have the same mean speed ω. Introducing an amplitude and a phase angle, we might write

$$\theta = \omega t + \frac{R}{I}\sin(\beta t + \epsilon), \quad R/I = \sqrt{(P^2 + Q^2)}, \quad \tan \epsilon = P/Q,$$

in which case we must write

$$\phi = \omega t + \frac{R}{J}\sin(\beta t + \epsilon + \pi).$$

Two conclusions can at once be drawn. The oscillations which the rotors perform about their mean speed are in opposite phase, so that the one is at the extremity of its lead when the other is at the extremity of its lag. This is shown by the π in the phase angle. And the factors I, J show that the amplitude for each rotor is inversely proportional to its inertia; the two angular displacements from the mean position preserve a constant ratio for all time.

The fact that the rotors oscillate in opposite directions suggests that we examine an intermediate point on the shaft. Choosing a point such that the left part is to the right part as λ to μ, the angular displacement there is

$$\frac{\mu\theta + \lambda\phi}{\lambda + \mu} = \omega t + R\sin(\beta t + \epsilon)\left(\frac{\mu}{I} - \frac{\lambda}{J}\right)/(\lambda + \mu).$$

There is no oscillation here if $\lambda/\mu = J/I$. Such a point is termed a node, and its position divides the distance in the inverse ratio of the inertiæ, just as the mass-centre of two masses divides the distance in the inverse ratio of the masses.

5, 2. Before proceeding with any further illustrations it is an advantage to recall a few simple properties of positive numbers. Let p, q be two such; then

(i) since $(p - q)^2$ is certainly not negative, we have

$$(p - q)^2 \geqslant 0,$$

whence
$$(p - q)^2 + 4pq \geqslant 4pq$$

or
$$(p + q)^2 \geqslant 4pq.$$

(ii) Consider two pairs p, q and x, y in which we may assume $p > q$ and $x > y$. Then if $pq = xy$ we cannot have $p > x$, $q > y$ or $p < x$, $q < y$. In other words, p and q cannot interlace x and y; either they are both inside the x, y interval or both outside. This is fairly obvious; but if we assume that p, q is the outside pair, so that $(p - q)$ is greater than $(x - y)$, it is not quite so obvious that $(p + q)$ is greater than $(x + y)$. As a numerical illustration let p, x, y, q be respectively 12, 9, 8, 6. The product $pq = 72 = xy$; p, q are the outside pair and their interval is 6, greater than the x, y interval which is unity. Their sum is $p + q = 18$, which is greater than that of the inner pair $x + y = 17$. The general proof is quite simple. We have, by hypothesis,

$$4pq = 4xy \quad \text{or} \quad (p + q)^2 - (p - q)^2 = (x + y)^2 - (x - y)^2,$$

and as
$$(p - q)^2 > (x - y)^2$$

it follows by addition that

$$(p + q)^2 > (x + y)^2 \quad \text{or} \quad (p + q) > (x + y).$$

(iii) Conversely, if $pq = xy$ and $p + q > x + y$, then $p - q > x - y$ and p, q are the outer pair.

(iv) If a pair of frequencies $(p, q)/2\pi$ lie outside another pair $(x, y)/2\pi$, then the corresponding periods $2\pi/(p, q)$ lie outside the other pair $2\pi/(x, y)$.

The propositions are useful in treating coupled systems with two frequencies. We may express them as:

(i) If two pairs of numbers have the same product, the pair with the greater difference is external and has the greater sum.

(ii) If two pairs of numbers have the same sum, the pair with the greater difference is external and has the smaller product.

(iii) Of two pairs with the same sum, the greater product is internal. It is instructive to view these propositions in conjunction with the rectangular hyperbola $xy =$ constant and the straight line $x + y =$ constant.

5, 3. *Coupled Systems.*

We can now continue with our examples, and as these have hitherto been limited by a single frequency we widen the scope. If a simple pendulum of length l carries a mass m, the period of oscillation is $2\pi/\omega_1$, where $\omega_1^2 = g/l$. We take a second simple pendulum of length h, mass M, period $2\pi/\omega_2$ where $\omega_2^2 = g/h$, and attach it to the mass m. We presume that this coupled system makes small oscillations in a vertical plane. If θ, ϕ are respectively the inclinations of l, h to the vertical at any instant, the corresponding lateral displacements of m and M are $x_1 = l\theta$ and $x_2 = l\theta + h\phi$. If T_1, T_2 are the tensions in l and h respectively, we have, in view of the small inclination to the vertical, $T_2 = Mg$ and $T_1 = (m + M)g$. The equations of horizontal motion are $M\ddot{x}_2 = -T_2\phi$ and $m\ddot{x}_1 = T_2\phi - T_1\theta$; otherwise

$$ml\ddot{\theta} = Mg\phi - (M + m)g\theta,$$
$$M(l\ddot{\theta} + h\ddot{\phi}) = - Mg\phi.$$

As these are homogeneous in the masses, we lighten the symbolism by putting $M/m = \mu$ and $\mu + 1 = n$. Accordingly n is necessarily greater than unity, and μ may have any positive value. The equations can now be written

$$(lD^2 + ng)\theta - \mu g\phi = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$
$$lD^2\theta + (hD^2 + g)\phi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

It is immaterial whether we eliminate θ or ϕ; in either case the symbolic equation is

$$(lD^2 + ng)(hD^2 + g) + \mu lgD^2 = 0.$$

On division by lh this becomes

$$D^4 + gnD^2\left(\frac{1}{l} + \frac{1}{h}\right) + \frac{ng^2}{lh} = 0.$$

which can be written

$$D^4 + nD^2(\omega_1^2 + \omega_2^2) + n\omega_1^2\omega_2^2 = 0.$$

The question then arises, what is the nature of the two roots of this quadratic in D^2? They are certainly not positive; nor are they pure imaginaries. They are not complex if

$$n^2(\omega_1^2 + \omega_2^2)^2 > 4n\omega_1^2\omega_2^2.$$

As n is greater than unity, this is equivalent to

$$(\omega_1^2 + \omega_2^2 \geqslant 4\omega_1^2\omega_2^2.$$

This is certainly true, and the roots must be negative. We may take $D^2 = -\alpha^2, -\beta^2$, and presume α the greater of the two. Incidentally, they cannot be equal since this would imply $n = 1$ and $\mu = 0 = M$.

Two new frequencies thus appear in the solution. They are $(\alpha, \beta)/2\pi$, and the question arises, how do they compare with the frequencies when each pendulum swings separately? As a matter of convenience we assume ω_1 to be greater than ω_2; it makes no difference except to our method of writing the argument. The theory of quadratics gives

$$\alpha^2\beta^2 = n\omega_1^2\omega_2^2 = (n\omega_1^2)\omega_2^2,$$

$$\alpha^2 + \beta^2 = n(\omega_1^2 + \omega_2^2) > n\omega_1^2 + \omega_2^2.$$

This decides that α^2, β^2 lie outside $n\omega_1^2$, ω_2^2, and hence certainly outside ω_1^2, ω_2^2; so that α, β are external to ω_1, ω_2. The subtraction of the equations (i), (ii) gives

$$ng\theta = (hD^2 + ng)\phi,$$

whence $$\theta = \left(1 + \frac{D^2}{n\omega_2^2}\right)\phi = \left(1 + \frac{\omega_1^2 D^2}{\alpha^2\beta^2}\right)\phi,$$

so that if we assume for ϕ the general solution

$$\phi = A\sin(\alpha t + \gamma) + B\sin(\beta t + \delta),$$

then $$\theta = A\left(1 - \frac{\omega_1^2}{\beta^2}\right)\sin(\alpha t + \gamma) + B\left(1 - \frac{\omega_1^2}{\alpha^2}\right)\sin(\beta t + \delta).$$

Of the two brackets that here multiply A and B, the former is essentially negative and the latter positive. The four constants A, B, γ, δ can be found when four consistent conditions are postulated, e.g. the initial displacements and velocities of the two masses. Two cases are outstanding. If A is zero, then θ, ϕ maintain a constant ratio for all time and have the same sign. If B is zero, the ratio θ/ϕ is still constant, but negative. These two motions are known as the normal modes; in general, a motion is compounded of the two. The conditions under which a normal mode of oscillation can be set up will be found among the exercises.

It is worth while drawing attention to a curious phenomenon in connexion with this motion. If the two pendulums are of nearly equal length, or $l = h$ approximately, then $\omega_1 = \omega_2$ approximately. We know that α cannot be equal to β; but if we make the upper mass relatively large, so that μ is small and $n = 1 + \mu$ is approximately unity, then α and β cannot be very disparate. If we further arrange that A, B are fairly close together, then ϕ is compounded of two simple harmonic motions of nearly equal amplitude and frequency. It is known that in such a case the excursion gradually changes from a maximum to a minimum and back again. The same thing happens to θ; but as its two components have different signs, θ is large when ϕ is small, and conversely. The kinetic energy of the motion seems gradually to transfer itself from the one mass to the other and back again. The phenomenon was first remarked by Euler in connexion with a swinging scale-pan.

As a second illustration of coupled systems we consider a simple series circuit, with a condenser of capacity C_1 and inductance L_1, and suppose I_1 is the current at any instant. We have previously seen that if the resistance is negligible, this circuit is oscillatory with a frequency given by $\omega_1{}^2 = 1/C_1L_1$. We take a second similar circuit, whose elements are distinguished by a different suffix, and couple them. If M denote the mutual inductance, we have, by Ohm's law,

$$R_1I_1 = V_1 - L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt},$$

so that, treating the resistance as negligible,

$$(1 + C_1L_1D^2)I_1 + C_1MD^2I_2 = 0.$$

Similarly,

$$(1 + C_2L_2D^2)I_2 + C_2MD^2I_1 = 0.$$

In strict theory $L_1L_2 = M^2$, but in practice M^2 is definitely less. We accordingly lighten the symbolism by writing $M/L_1 = \lambda_1$, $M/L_2 = \lambda_2$, so that $\lambda_1\lambda_2$ is less than unity, and we can write $\lambda_1\lambda_2 = 1 - n$, where n is small and positive. We divide the first equation by C_1L_1, and have

$$(\omega_1{}^2 + D^2)I_1 + \lambda_1D^2I_2 = 0.$$

Similarly,

$$(\omega_2{}^2 + D^2)I_2 + \lambda_2D^2I_1 = 0.$$

The elimination of I_1 or I_2 gives

$$(\omega_1^2 + D^2)(\omega_2^2 + D^2) = (1 - n)D^4$$

or

$$nD^4 + D^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 = 0.$$

Proceeding as before, in regarding this as a quadratic in D^2, the two roots are certainly not positive, nor are they pure imaginaries. Moreover, as $(\omega_1^2 + \omega_2^2)^2$ is not less than $4\omega_1^2\omega_2^2$, it is definitely greater than $4n\omega_1^2\omega_2^2$. Hence the roots cannot be equal or complex. We accordingly take them to be $D^2 = -\alpha^2, -\beta^2$, where α may be regarded as greater than β. Assuming for convenience that ω_1 is greater than ω_2, we have, by the theory of quadratics,

$$\alpha^2 + \beta^2 = \frac{\omega_1^2}{n} + \frac{\omega_2^2}{n} > \frac{\omega_1^2}{n} + \omega_2^2,$$

$$\alpha^2\beta^2 = \frac{\omega_1^2\omega_2^2}{n} = \frac{\omega_1^2}{n} \cdot \omega_2^2.$$

Hence α^2, β^2 are external to ω_1^2/n, ω_2^2; still more so are α, β external to ω_1, ω_2. If we eliminate D^2I_2, we get

$$(\omega_1^2 + nD^2)I_1 = \lambda_1\omega_2^2I_2,$$

so that if we choose for I_1 the general solution

$$I_1 = A\sin(\alpha t + \gamma) + B\sin(\beta t + \delta)$$

we have

$$\lambda_1\omega_2^2I_2 = A(\omega_1^2 - n\alpha^2)\sin(\alpha t + \gamma) + B(\omega_1^2 - n\beta^2)\sin(\beta t + \delta).$$

Here again the bracket factors of A and B are one positive and the other negative, and we again find that the frequencies of the normal modes of a coupled system fall outside the free frequencies of the separate systems. The parallel discussion when resistance or damping is present is beset with considerable difficulties.

5, 4. *Application to Dynamics.*

Simultaneous equations make a natural appearance in solving dynamical problems by the use of Lagrange's equations. If a conservative system has kinetic energy T and potential energy V, the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q}.$$

Here q is any one of the generalized co-ordinates and $\dot{q}$ is the corresponding generalized velocity. There is one such equation for each co-ordinate, and the total equals the number of degrees of freedom possessed by the system. A first integral necessarily exists in the integral of energy, and occasionally the principle of linear or angular momentum may supply another. The equations are of the second order with non-constant coefficients and are rarely soluble. It is seldom stated in print, but is none the less a fact, that the vast majority of dynamical problems are intractable; the soluble cases find their way into textbooks for students to practise on.

A simplification occurs when discussing small oscillations about an equilibrium position. It is known that in this case the potential energy, as measured from the equilibrium position, is a homogeneous quadratic in the co-ordinates with constant coefficients. Similarly, the kinetic energy is a homogeneous quadratic in the velocities with constant coefficients. Limiting ourselves to two co-ordinates for the sake of illustration, this gives

$$2T = a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2,$$

$$2V = b_{11}q_1^2 + 2b_{12}q_1q_2 + b_{22}q_2^2.$$

The corresponding Lagrangian equations are

$$a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + b_{11}q_1 + b_{12}q_2 = 0,$$

$$a_{12}\ddot{q}_1 + a_{22}\ddot{q}_2 + b_{12}q_1 + b_{22}q_2 = 0.$$

These can be written symbolically as

$$(a_{11}D^2 + b_{11})q_1 + (a_{12}D^2 + b_{12})q_2 = 0,$$

$$(a_{12}D^2 + b_{12})q_1 + (a_{22}D^2 + b_{22})q_2 = 0.$$

Such a system is necessarily soluble in exponentials. If $D^2 = -\lambda$, we have partial solutions of the type $A\cos(t\sqrt{\lambda} + \phi)$, where λ is a root of the equation

$$\begin{vmatrix} (a_{11}\lambda - b_{11}) & (a_{12}\lambda - b_{12}) \\ (a_{12}\lambda - b_{12}) & (a_{22}\lambda - b_{22}) \end{vmatrix} = 0$$

formed by eliminating one of the co-ordinates.

The kinetic energy is necessarily positive; or to adopt the approved terminology, the form $2T$ is "positive definite", whatever the number of co-ordinates. It is then known, from a theorem due to

Sylvester, that the zeros of the corresponding determinant in λ are all real. If the form for the potential energy $2V$ is also positive definite, which means dynamically that the system is stable in all co-ordinates, then it is known from algebra that the values of λ are all positive.

Still limiting ourselves to the case of two co-ordinates, we therefore have the solutions

$$q_1 = A_1 \cos(t\sqrt{\lambda_1} + \alpha_1) + A_2 \cos(t\sqrt{\lambda_2} + \alpha_2),$$
$$q_2 = B_1 \cos(t\sqrt{\lambda_1} + \beta_1) + B_2 \cos(t\sqrt{\lambda_2} + \beta_2).$$

Of the eight arbitrary constants which appear here, only four are independent. In general, when there are n co-ordinates there are $2n$ arbitrary parameters. They are determined by specified initial values of the co-ordinates and velocities.

EXERCISES

1. Solve the equations

$$\frac{dx}{dt} + 5x + 2y = 0,$$
$$\frac{dx}{dt} + \frac{dy}{dt} + 5y = 0,$$

subject to the initial conditions

$$t = 0 = y, \quad x = 3.$$

[$x = (3\cos 3t - \sin 3t)e^{-4t}, \quad y = 5e^{-4t}\sin 3t.$]

2. $(D + 5)x - 2y = e^t,$
$(D + 6)y - x = e^{2t}.$

$$\left[x = \frac{e^{2t}}{27} + \frac{7e^t}{40} + Ae^{-4t} + Be^{-7t}, \quad y = \frac{7e^{2t}}{54} + \frac{e^t}{40} + \tfrac{1}{2}Ae^{-4t} - Be^{-7t}.\right]$$

3. If the motion of a point x, y is governed by the equations

$$\frac{d}{dt}(x + y) = y = x + 4,$$

prove that the point can never pass through the origin.

[$x = ae^{\frac{1}{2}t} - 4, \quad y = ae^{\frac{1}{2}t}.$]

4. The co-ordinates of a moving point satisfy the equations

$$\frac{d^2x}{dt^2} + 3x - 4y = 3,$$
$$\frac{d^2y}{dt^2} + x - y = 5.$$

Deduce that $2y - x$ has the form $7 + a\cos t + b\sin t$.

5. A particle is initially at rest at the origin. Its motion is given by the equations

$$\frac{d^2x}{dt^2} = a - b\frac{dy}{dt}, \quad \frac{d^2y}{dt^2} = b\frac{dx}{dt}.$$

Prove that the initial motion must be along OX, and that the co-ordinates are $b^2x = a(1 - \cos bt)$, $b^2y = a(bt - \sin bt)$. Equations of this type occur in electron theory.

6. Equations of the type

$$\frac{d^2y}{dt^2} + \omega^2 y = c\frac{dx}{dt},$$

$$\frac{d^2x}{dt^2} + \omega^2 x = -c\frac{dy}{dt},$$

occur in the discussion of the Zeeman effect. Prove that the operator D necessarily takes a purely imaginary value.

Assuming that $x = Ae^{int}$, $y = Be^{int}$, prove by substitution that $B = \pm iA$.

Putting $z = x + iy$, show that the two equations can be written jointly as

$$(D^2 - icD + \omega^2)z = 0.$$

Deduce that a possible solution is circular motion with constant angular velocity.

7. In analogy with the two rotors on the same shaft, discuss the problem of two masses on the same spring. The spring, of stiffness k, has masses M, m attached to the left, right respectively. It is given an extension c; the system is placed at rest on a smooth horizontal table and left free to move. In the subsequent motion, assume that M, m have displacements x, y respectively to the right of their initial positions, so that the extension is $(c - x + y)$. Write down the equation for each mass; interpret their sum and its first and second integrals. Eliminate y and simplify by writing $k/M = \omega_1^2$, $k/m = \omega_2^2$; these give the free periods of the separate masses. The general value of x is $A + Bt + P\cos nt + Q\sin nt$, where $n^2 = \omega_1^2 + \omega_2^2$. The initial conditions lead to $B = 0 = Q$, and you should reach the result

$$x = \frac{c\omega_1^2}{n^2}(1 - \cos nt), \quad y = -\frac{c\omega_2^2}{n^2}(1 - \cos nt).$$

The inference is that the masses are always moving in opposite directions. They perform simple harmonic motions of equal frequency, the amplitudes being inversely proportional to the masses. The node is their mass-centre.

8. In the case of the coupled oscillating circuits discussed in the text, assuming that the separate circuits have the same capacity and the same natural frequency, prove that the periods of the coupled system are given by

$$\frac{1}{\alpha^2} = C_1(L_1 - M), \quad \frac{1}{\beta^2} = C_1(L_1 + M).$$

9. In connexion with the stability of an airship we meet the equations

$$\eta'' + a\eta' - b\theta' - c\theta = 0,$$

$$\theta'' - p\theta' - q\theta + \frac{qa}{c}\eta' = 0,$$

where the coefficients are all constants. Prove that a possible form for η is

$$Ae^{\alpha t} + Be^{\beta t} + Ct + F,$$

in which case θ has the form

$$Pe^{\alpha t} + Qe^{\beta t} + \frac{aC}{c}.$$

The important difference is that unless C is zero, η must increase indefinitely on account of the secular term Ct. The same does not apply to θ.

10. In the equations

$$(D^2 + a^2)x + bDy = 0,$$
$$(D^2 + c^2)y - dDx = 0,$$

where all the constants are positive, prove that x and y are necessarily combinations of undamped oscillations.

11. When the two rotors on the same shaft are a ship's propeller and the rotary parts of the engine, a first approximation is

$$I\ddot{\theta} + C(\theta - \varphi) = A \sin \omega t,$$
$$J\ddot{\varphi} + b\dot{\varphi} = C(\theta - \varphi).$$

This allows for a variable crank-effort and for propeller-resistance varying with the speed. Presuming that there is no resonance, prove that θ and φ have particular integrals of the form $R \sin(\omega t + \varepsilon)$, and that the complementary functions are pretty certainly of the form

$$E + Fe^{-\gamma t} + Ge^{-\alpha t} \sin(\beta t + \delta).$$

The first two terms are not in doubt.

12. Establish the following results in connexion with the double pendulum: (i) The quicker normal mode can be established by starting the masses with no initial displacements, the initial velocities being in opposite directions and in the ratio $\dot{\varphi}/\dot{\theta} = \beta^2/(\omega_1^2 - \beta^2)$. In this case the masses are always moving in opposite directions. Their displacements keep a constant ratio; so do their velocities.

(ii) The slower mode can be started from rest if the initial displacements have the ratio $\varphi/\theta = \alpha^2/(\alpha^2 - \omega_1^2)$. In this case the masses are always moving in like directions. Their displacements keep a constant ratio; so do their velocities.

In the case of the energy transfer, verify the statement in the text that θ is large when φ is small. Prove also that the maximum amplitude of φ is large compared with that of θ. Incidentally, this last statement can be rendered highly probable from first principles.

CHAPTER VI

Fourier Series

6, 1. A competent mathematical student might easily spend several months learning what is known of Fourier series. At the end of that time he would find that a good deal remained to be cleared up. Needless to say, the present chapter has no pretensions to completeness, and like all unspecialized expositions it demands a good deal of uncritical acceptance. The justification for its presence is that some slight knowledge of Fourier series is necessary for a comprehension of partial differential equations even in their elementary applications. The presentation is reduced to a minimum consistent with adequacy; there are several more lengthy expositions available.

6, 1·1. Jean Baptiste Fourier was a French physicist who went to Egypt with Napoleon's army. Like Gaspard Monge, the founder of descriptive geometry, he was left behind when Napoleon withdrew. In 1807 he startled the Paris Academy with the heretical proposition that a finite, one-valued function $f(x)$ could be expressed over a finite range by a series of the form

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + b_1 \sin x + b_2 \sin 2x + \dots,$$

a series which nowadays universally bears his name. Lagrange was frankly incredulous, and ever since then mathematicians have been arguing as to how far Fourier was right. The argument still goes on; meanwhile less sophisticated people see in Fourier's theorem a tool that does the job and accept it as such. The three main questions that naturally present themselves are:

(*i*) How the coefficients are to be determined for a given $f(x)$.

(ii) Whether the resulting series will necessarily be convergent (otherwise it has little meaning).

(iii) Whether the series, if convergent, will converge to the corresponding value of $f(x)$ for all admissible values of x (otherwise it is not a strict representation of $f(x)$).

We leave the second and third of these questions to be answered in the affirmative by the mathematicians. The first then presents little difficulty if we have no scruples about applying term-by-term integration to a series; the matter is dealt with later.

6, 2. To get the idea, erect two ordinates on OX, as far apart or as close together as you like. This is the "finite range" mentioned above; in practice it may be anything from the thickness of a boiler tube to the length of a submarine cable. Passing from one of these ordinates to the other, you draw any sort of graph you like. It need not be a continuous curve; parts of it may look as if they had dropped out of place, but there are one or two restrictions (see fig. 1). In the first place it must nowhere go up to infinity, like the graph of $\tan x$. Nor must it double back on itself and risk being cut more than once by the same ordinate. Horizontally there must be no gaps (otherwise

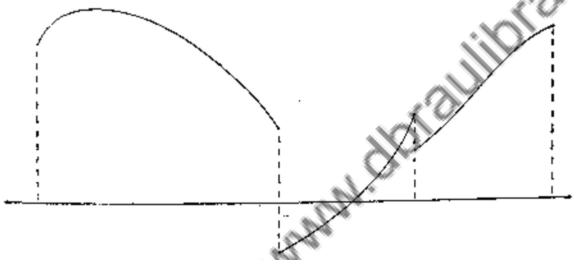

Fig. 1

there is no value for the ordinate); nor any overlapping of separate parts (otherwise we get more than one value for certain ordinates). Vertical gaps, corresponding to discontinuities, are permitted. So are negative and zero values; part of OX may be included. This is the one-valued function mentioned above, and the contention is that the equation of the "graph" can be expressed by a Fourier series.

A possible objection is that, at a discontinuity, the ordinate is met twice. A sufficient answer is: omit the last point of the one part or the first point of the other, and then ask yourself how much difference that is likely to make to the result. Better still, split the difference by omitting both, and take their mid-point as a substitute for the pair; because that is exactly what theory (the theory which we shall not discuss) indicates as the proper thing to do. You will find it written in the treatises as $\frac{1}{2}[f(x+0)+f(x-0)]$, which is exactly what has just been said.

It is hardly surprising that the magnitude of the range is of no consequence so long as it is finite; for the number that measures it might have any magnitude, according to the unit of measurement.

We take it as 2π, and consider the range as extending either from $-\pi$ to $+\pi$ or from 0 to 2π; there are certain slight advantages in the former.

6, 3. Looking at the Fourier series, we can see that if x be changed by 2π (or for that matter by any positive or negative integral multiple of 2π), each term retains its value. Accordingly, if we draw the graph of the series, then whatever it looks like between 0 and 2π, it looks exactly the same between 2π and 4π, or 4π and 6π; and so on, left and right. In other words, it gives a repeating pattern, such as might be a dog-tooth moulding or a scalloped edge. On the other hand, the function we are working with, maybe a bending moment diagram, may cease to exist outside the range; or if we are working with a vertical parabola, nose downwards, between $x = \pi$ and $x = -\pi$, it may go to infinity outside the range, whereas the graph of the corresponding Fourier series would resemble a chain hung from a line of equidistant posts. The point is that inside the range we can make the Fourier series fit as closely as we like to any figure we like by taking a sufficient number of terms. What happens outside the range is not our concern, except that we must not attempt to use the Fourier series outside the range for which it was formed.

Having decided to accept the proposition as part of our equipment, it is not difficult to find how to use it, especially if we prepare the ground a little in advance. We begin with a word on periodic functions. If $f(x) = f(x + \beta)$ for all values of x, then $f(x)$ is said to be a periodic function and β is called a period. Since the statement is true by hypothesis when x is replaced by $(x + \beta)$ or $(x - \beta)$, we have

$$f(x - \beta) = f(x) = f(x + \beta) = f(x + 2\beta) \ldots .$$

If β is the smallest number for which this is true, then β is called "the period". As common examples, $\sin x$ is a periodic function of period 2π; $\cos^2 2x$ is a periodic function with period $\frac{1}{2}\pi$.

As to changing the period, if we substitute $x = nX$, we have

$$f(nX) = f(nX + \beta) = f\left\{n\left(X + \frac{\beta}{n}\right)\right\}.$$

The only difference between the first and last of these forms is that where the one contains X the other contains $(X + \beta/n)$. Hence, if we call the first $F(X)$, then we must call the last $F(X + \beta/n)$, and we have

$$F(X) = F\left(X + \frac{\beta}{n}\right).$$

The period is now β/n, and this can be made to take any value by properly choosing n. The case that interests us most is where β is the 2π of theory and β/n is the λ of practice, being the length of a girder maybe. With $\beta = 2\pi$ and $\beta/n = \lambda$, we have $n = 2\pi/\lambda$. Hence the rule: to change a function from period 2π to period λ, replace x by $2\pi x/\lambda$. Conversely, to change from period λ to 2π, replace x by $\lambda x/2\pi$.

6, 4. *Some Definite Integrals.*

We shall have to make frequent use of certain definite integrals. If m, n are any two positive integers, it is easily shown that

$$\int_0^{2\pi} \cos mx\, dx = 0 = \int_0^{2\pi} \sin nx\, dx.$$

The results are in fact intuitive from the graphs of $y = \cos mx$ and $y = \sin nx$. The other integrals which we shall require are all contained in the expression

$$\int_{-\pi}^{\pi} \begin{matrix}\sin\\ \cos\end{matrix}\, mx\, \begin{matrix}\sin\\ \cos\end{matrix}\, nx\, dx.$$

The verification or acceptance of the following statements is left to the reader: (i) If the integers are different, the four results are all zero. (ii) If the functions are different, the result is zero. (iii) The only time we get a non-zero result is when the functions are the same and the integers are the same. Then we have

$$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi = \int_{-\pi}^{\pi} \sin^2 nx\, dx.$$

6, 5. *Determination of the Coefficients.*

We can now approach the proposition that in the range $-\pi < x < +\pi$ we have

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + \dots .$$

It is international usage and almost traditional to take b as the coefficient of sine, and a as the coefficient of cosine. The suffix agrees with the multiple of x. One slight variation is that theory prefers to use $\frac{1}{2}a_0$ for the independent term.

The first question is how the coefficients are to be determined. Any one of them will serve as an illustration, so we take a_3, which multiplies $\cos 3x$. Multiply both sides by $\cos 3x$ and integrate from $-\pi$ to $+\pi$. In accordance with the integrals just mentioned, every term on the right gives zero, with a single exception. We have

$$\int_{-\pi}^{\pi} f(x)\cos 3x\,dx = a_3\int_{-\pi}^{\pi}\cos^2 3x\,dx = \pi a_3.$$

Another coefficient, such as b_4, would be obtained by multiplying both sides by the co-factor of b_4, i.e. $\sin 4x$ and integrating from $-\pi$ to $+\pi$. Hence, in general,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

The absolute term a_0 is naturally an exception. To find its value we integrate the equation as it stands. Every term on the right, except the first, gives zero, and we have

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\int_{-\pi}^{\pi} dx = 2\pi a_0.$$

This interprets a_0; as 2π is the base and the left side is the area, we see that a_0 is the mean ordinate; a useful piece of information, as it can frequently be written down at sight.

6, 6. *A Useful Application.*

At this stage we can consider the problem of finding the Fourier representation of that part of the parabola $y = x^2$ that lies in the range $-\pi < x < \pi$. We have

$$x^2 = a_0 + a_1\cos x + a_2\cos 2x + \ldots$$
$$+ b_1\sin x + b_2\sin 2x + \ldots,$$

and the problem is to determine the coefficients. If the mean ordinate cannot be written down at sight (though it is fairly common knowledge that it must be one-third of the terminal ordinate), we have

$$2\pi a_0 = \int_{-\pi}^{\pi} x^2\,dx = 2\pi^3/3$$

and $a_0 = \pi^2/3$. For the other coefficients we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx.$$

These can be evaluated by repeated integration by parts. It is probably quicker to obtain them by differentiating twice with respect to n the indefinite integrals of $\cos nx$ and $\sin nx$. This gives

$$-\int \cos nx\,dx = -\frac{\sin nx}{n}; \quad -\int \sin nx\,dx = \frac{\cos nx}{n},$$

$$\int x \sin nx\,dx = -\frac{x\cos nx}{n} + \frac{\sin nx}{n^2};$$

$$-\int x \cos nx\,dx = -\frac{x \sin nx}{n} - \frac{\cos nx}{n^2},$$

$$\int x^2 \cos nx\,dx = \frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3},$$

$$\int x^2 \sin nx\,dx = -\frac{x^2\cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2\cos nx}{n^3}.$$

The insertion of the limits gives

$$\int_{-\pi}^{\pi} x^2 \sin nx\,dx = 0; \quad \int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{4\pi \cos n\pi}{n^2}.$$

It appears from this that all the b's are zero, and there are no sine terms. It will be seen later that this could have been stated at the outset. The value of the coefficient a_n is $(4\cos n\pi)/n^2$, so that by giving n successive integer values we have

$$a_1 = -4/1^2, \quad a_2 = +4/2^2, \quad a_3 = -4/3^2,$$

and so on. Hence

$$x^2 = \frac{\pi^2}{3} - 4\left\{\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\right\}. \qquad \text{(i)}$$

This is the Fourier series which represents the parabola $y = x^2$ in the range $-\pi < x < \pi$. At the risk of being tedious we again point out that, outside the range, the parabola and its representation part company. The parabola goes to infinity, whereas the graph of the Fourier series looks like festoons left and right (fig. 2). Inside the range the fit can be made as close as we like by taking a sufficient number of terms.

The ordinate at each end of the range is π^2. We can lift the axis OX through this distance by putting $y_1 = y - \pi^2$. The figure now looks like a parabolic channel with OX_1 as ground level. The width of the channel, at present 2π, can be changed to λ by replacing x by

$2\pi x/\lambda$. Notice that this merely alters the horizontal scale of drawing; the depth is unchanged. We can alter the central depth from π^2 to h by putting $y_2 = hy_1/\pi^2$. Hence

$$y_2 = \frac{h}{\pi^2}\, y_1 = \frac{h}{\pi^2}\,(y - \pi^2)$$

gives

$$-\frac{\pi^2}{h}\, y_2 = \frac{2\pi^2}{3} + 4\left\{\frac{1}{1^2}\cos\frac{2\pi x}{\lambda} - \frac{1}{2^2}\cos\frac{4\pi x}{\lambda} + \frac{1}{3^2}\cos\frac{6\pi x}{\lambda} - \ldots\right\}.$$

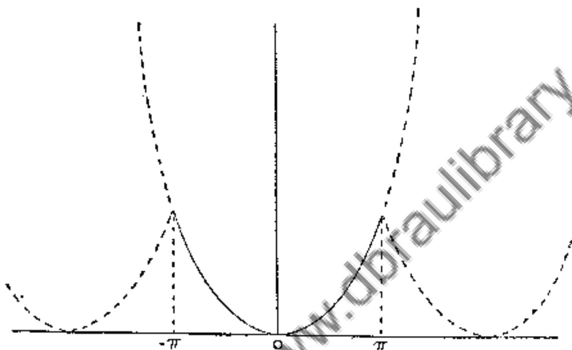

Fig. 2

We now have a representation of a parabolic channel of width λ and central depth h. By changing the sign of y_2, i.e. inverting the figure, we have a parabolic arch of span λ and rise h, the axes being ground-level and centre-line. If we then replace h by $w\lambda^2/8$, we have the bending moment diagram of a simply supported beam, of length λ, bearing uniform load w per unit length. The fact that a bending-moment diagram is conventionally drawn positive downwards makes no difference.

$$y = \frac{w\lambda^2}{12} + \frac{w\lambda^2}{2\pi^2}\left\{\frac{1}{1^2}\cos\frac{2\pi x}{\lambda} - \frac{1}{2^2}\cos\frac{4\pi x}{\lambda} + \ldots\right\}.$$

6, 7. *Simplifications.*

It begins to appear likely that a problem in Fourier series involves a good deal of integration. That usually is the case; fortunately, the labour can frequently be economized by a few simple observations. Consider a curve which is symmetrical about OY, so that the left side is the reflection in OY of the right side; written algebraically, $F(x) = F(-x)$. If we need the area between $x = c$ and $x = -c$, we can evidently take the right half and double it. Hence for a symmetrical curve $y = F(x)$ we have

$$\int_{-c}^{c} y\,dx = 2\int_{0}^{c} y\,dx.$$

A simple example of a symmetrical curve is $y = \cos x$.

If we now reflect the left half of our symmetrical curve in OX and use the parts in the first and third quadrants, we have a function which is said to be skew; written algebraically, $F(x) = -F(-x)$. A simple example is $y = \sin x$. The area of any such skew curve taken between $x = c$ and $x = -c$ is evidently zero. It is easily proved that:

(i) the product of two symmetrical functions is symmetrical;

(ii) the product of two skew functions is symmetrical;

(iii) the product of a skew function and a symmetrical function is skew.

This terminology is not universal. The functions are frequently referred to as even and odd; but these adjectives are later required with a different connotation and experience shows that skew and symmetrical are preferable.

Certain important consequences immediately follow from this. Suppose we happen to be working with a symmetrical function $y = f(x)$. When we attempt to find the sine terms, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

The integrand is the product of $f(x)$, which is given as symmetrical, and $\sin nx$ which is skew. The integrand is therefore skew and b_n is zero. There are no sines. Hence the important result: *a symmetrical function can be expressed in cosines only, and a function expressed in cosines is symmetrical.*

This substantiates the remark in 6, 6 about the representation of $y = x^2$. The function is symmetrical, and hence the Fourier series contains only cosines. In the same way it follows that *a skew function can be expressed in sines only; and a function expressed in sines is skew.*

When the function is skew and we determine b_n as above, we have $f(x)$ skew and $\sin nx$ skew. Hence the integrand is symmetrical. We can therefore double the integral over the half-range and write

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx.$$

A word of caution is desirable here; whenever this form is used, the $f(x)$ must be taken at its value in the integration range 0 to π. This is important because the mathematical expression of $f(x)$ is frequently different in different parts of the range. It is a simple exercise for the reader to prove similarly that if $f(x)$ is symmetrical, we can write

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx.$$

We can illustrate this last formula by our parabola. We have

$$a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^\pi.$$

Here the middle term is the only one that survives the substitution of the limits and we have $a_n = (4\cos n\pi)/n^2$ as before.

The possibility of calculating the coefficients by doubling the integral over the half-range is not confined to skew and symmetric functions; it applies to two other types as well. Consider a function $F(x)$, defined in the range $-\pi < x < \pi$, in which the right half is a reproduction of the left half, so that if the left half were displaced bodily a distance π to the right the two halves would coincide. The period is now really π, and mathematically we have

$$F(x) = F(x-\pi) = F(x+\pi).$$

Incidentally, if n is even, $\cos nx$ and $\sin nx$ are of this type, which we may call *Type I*. As an example we have the function defined by

$$y = e^{-x}, \qquad 0 < x < \pi,$$

$$y = e^{-(x+\pi)}, \quad -\pi < x < 0.$$

In such a case we evidently have

$$\int_{-\pi}^{\pi} F(x)\,dx = 2\int_0^\pi F(x)\,dx.$$

Alternatively, consider a function $F(x)$ in which the right half is a reproduction of the left half but inverted, so that if the left half were displaced bodily a distance π to the right, the two halves would be mirrored in OX. We now have

$$F(x) = -F(x-\pi) = -F(x+\pi),$$

and if n is odd, then $\cos nx$ and $\sin nx$ are both of this *Type II*. Evidently the integral of such a function, taken over the range $-\pi$ to π, is zero. An example is the function defined by (*vide infra*, fig. 3)

$$y = e^{-x}, \qquad 0 < x < \pi,$$
$$y = -e^{-(x+\pi)}, \quad -\pi < x < 0.$$

We can now state that two functions of the same type have a product of Type I; two functions of different types have a product of Type II.

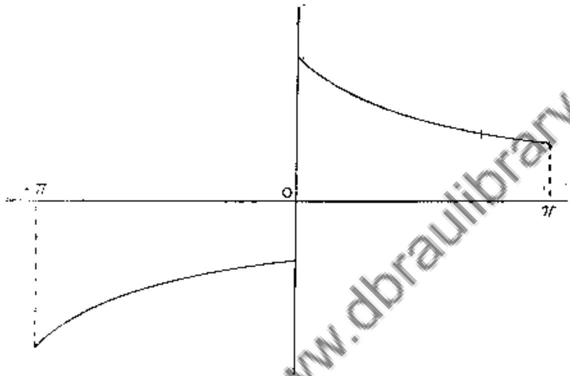

Fig. 3

The proofs are quite simple. If $F_1(x)$, $F_2(x)$ are two functions of Type I, we have, by hypothesis,

$$F_1(x) = F_1(x - \pi), \quad F_2(x) = F_2(x - \pi).$$

Hence
$$F_1(x)F_2(x) = F_1(x - \pi)F_2(x - \pi)$$
or
$$F(x) = F(x - \pi),$$

so that the product is of Type I. Alternatively, if both functions are of Type II, we have, by hypothesis,

$$F_1(x) = -F_1(x - \pi), \quad F_2(x) = -F_2(x - \pi),$$

whence
$$F_1(x)F_2(x) = +F_1(x - \pi)F_2(x - \pi)$$
or
$$F(x) = F(x - \pi),$$

so that the product is again of Type I. This establishes the first statement; the second can be proved analogously.

We apply these principles to the determination of the Fourier coefficients and begin by supposing that $f(x)$ is of Type I. The integrand for a_n or b_n is of Type II if n is odd, so that the integral is zero. Alternatively, if n is even, the integrand is of Type [illegible], and the integral can be doubled over the half-range. Hence the important result:

If $f(x) = f(x - \pi)$ there are no coefficients of odd suffix; the coefficients of even suffix can be determined by doubling the integral over the half-range. Such a function is said to be expressible in "even harmonics".

In a similar manner, by making obvious changes in the wording, we prove that:

If $f(x) = -f(x - \pi)$ there are no coefficients of even suffix; the coefficients of odd suffix can be determined by doubling the integral over the half-range. Such a function is said to be expressible in odd harmonics. In this case a_0 is zero.

As an example we consider (fig. 3) the function mentioned previously and defined by

$$y = e^{-x}, \qquad 0 < x < \pi,$$
$$y = -e^{-(x+\pi)}, \quad -\pi < x < 0.$$

This fulfils the conditions for being expressible in odd harmonics and we have

$$f(x) = a_1 \cos x + a_3 \cos 3x + a_5 \cos 5x \ldots$$
$$+ b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x \ldots$$

where

$$\pi a_n = 2\int_0^\pi f(x) \cos nx\,dx \qquad \pi b_n = 2\int_0^\pi f(x) \sin nx\,dx$$
$$= 2\int_0^\pi e^{-x} \cos nx\,dx \qquad = 2\int_0^\pi e^{-x} \sin nx\,dx$$
$$= \frac{2(1 + e^{-\pi})}{(n^2 + 1)}, \qquad = \frac{2n(1 + e^{-\pi})}{(n^2 + 1)}.$$

It is to be observed that the value given to $f(x)$ in the integrand is its value in the range of integration 0 to π. We completely ignore the fact that its value in the other half-range is differently expressed. This gives the Fourier series for the function as

$$\frac{\pi y}{2(1 + e^{-\pi})} = \Sigma \frac{\cos nx + n \sin nx}{n^2 + 1}, \quad n \text{ odd.} \quad \ldots \quad \text{(i)}$$

Here we may draw attention to a minor point that sometimes harasses students. If we give n an even value in the foregoing calculation of the coefficients, the result is not zero. This sometimes occasions surprise, seeing that the coefficients of even suffix are known to be zero. The explanation is that both integrand and limits have been modified on the assumption that n is odd; if we invalidate this hypothesis, it is hardly to be wondered at that we get an unexpected result.

The fact that a function may be expressible in even harmonics only, or odd harmonics only, coupled with the fact that it may be skew or symmetrical, leads to four possibilities. A function may be expressible in (i) even cosines; (ii) even sines; (iii) odd cosines; (iv) odd sines. In each of these cases the coefficients can be calculated by taking four times the integral over the quarter-range. The reader should have no difficulty in convincing himself, both algebraically and graphically, that a function which comes under a specified one of these four possibilities is defined in the full range when it is defined in a quarter-range.

To prove the statement we begin by considering an $F(x)$ which is symmetrical and of Type I, i.e. expressible in even cosine harmonics, since ultimately all four cases reduce to this. The function by hypothesis has the properties

$$F(-x) = F(x) = F(x-\pi) = F(\pi - x)$$

for all values of x. On replacing x by $(\frac{1}{2}\pi - x)$, we have

$$F(\tfrac{1}{2}\pi - x) = F(\tfrac{1}{2}\pi + x),$$

which proves that $F(x)$ is symmetrical about $x = \frac{1}{2}\pi$ as well as about $x = 0$. Hence

$$\int_0^{\frac{1}{2}\pi} F(x)\,dx = \int_{\frac{1}{2}\pi}^{\pi} F(x)\,dx = \tfrac{1}{2}\int_0^{\pi} F(x)\,dx$$

and $$\int_{-\pi}^{\pi} F(x)\,dx = 2\int_0^{\pi} F(x)\,dx = 4\int_0^{\frac{1}{2}\pi} F(x)\,dx.$$

Let us apply this to the calculation of the coefficients in any one of the aforementioned cases, say case (iv), odd sines. We have by hypothesis

$$f(x) = b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \dots .$$

The determination of b_n depends on integrating $f(x)\sin nx$, with n odd. Both factors of this integrand are skew and of Type II; hence their product is symmetrical and of Type I, like the $F(x)$ mentioned above.

Hence $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} f(x)\sin nx\,dx.$$

The proof of the other three cases is exactly similar, and can be left to the reader.

As an example, consider the function whose graph is a repetition of the graph of $\cos x$ in the range $-\pi < x < 0$. Its definition is

$$y = \cos x, \quad -\pi < x < 0,$$
$$y = -\cos x, \quad 0 < x < \pi.$$

Fig. 4 shows it to be skew and of Type I; hence

$$y = b_2 \sin 2x + b_4 \sin 4x + b_6 \sin 6x + \ldots .$$

We have
$$b_n = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} f(x) \sin nx\, dx$$

$$= -\frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \cos x \sin nx\, dx = -\frac{4n}{(n^2-1)\pi}.$$

Thus
$$y = -\frac{4}{\pi}\left\{\frac{2}{1.3}\sin 2x + \frac{4}{3.5}\sin 4x + \ldots\right\}.$$

This last example illustrates a point in theory. The function is discontinuous when $x = 0$; the left half-range gives the corresponding value $y = 1$, whereas the right half-range gives -1. Theory indicates that the series will give neither value, but a value midway between. This is evidently correct.

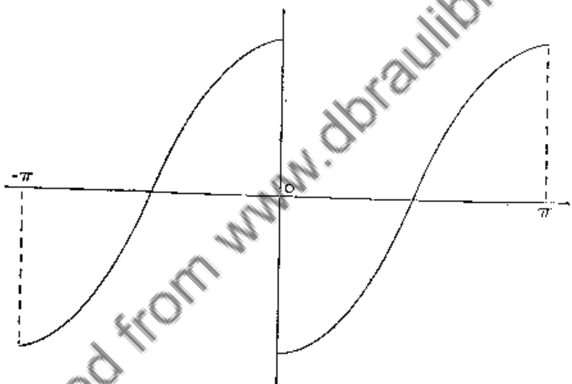

Fig. 4

When there is no discernible symmetry or regularity in the given function, there is no means of economizing the labour of calculation, and each coefficient has to be separately worked out. If the function is variously defined in different parts of the range and each coefficient depends on more than one integral, the work can become very tedious if carried to the higher harmonics. Thus, if

$$y = f_1(x), \quad -\pi < x < c,$$

$$y = f_2(x), \quad c < x < \pi,$$

we have
$$\pi a_n = \int_{-\pi}^{c} f_1(x) \cos nx\, dx + \int_c^{\pi} f_2(x) \cos nx\, dx,$$

and similarly for the other coefficients.

6, 8. *Analysis in a Prescribed Mode.*

It is frequently desirable in practice to analyse a given function in some specified way, such as even harmonics only; or maybe odd sines. When the given function has a single specification, viz. (i) cosines, (ii) sines, (iii) odd harmonics, or (iv) even harmonics, we regard the function as being defined in the right half-range. The left half-range is then filled in, either mentally or graphically, in accordance with the specification, i.e. symmetrically for cosines, skew for sines, and so on. The student should take some simple graph, say $y = e^{-x}$ in the range 0, 1, and complete the figure for analysis in each of the four given modes; this makes the total range equal to 2.

When the required mode of analysis has a double specification, e.g. odd cosine harmonics, the given function is considered as filling the first quarter-range to the right of the origin. The rest of the range is then filled in to comply with the specification. Taking the previous example of $y = e^{-x}$ in the range 0, 1, the reader should have no difficulty in convincing himself that the specification "even sine harmonics" is not different from "sines only"; and the same is true reading cosine for sine.

The conditions of a problem sometimes dictate the mode of analysis. Thus, if a horizontal beam be uniformly loaded and simply supported at quarter-span from each end, the shearing force diagram must be even sines if the origin is the middle of the beam.

In nearly all such problems as occur in practice the period is not the 2π of theory. This poses a dilemma: shall we alter the rules for calculating the coefficients so as to fit the new range, or shall we alter the range (and the function with it) so as to comply with the established rules? The author has no misgivings about the second course being preferable, especially as the rules for altering the period have already been examined. Sometimes it is even better to alter both the vertical and horizontal scale for the sake of getting a simpler arrangement: this has already been exemplified by using the parabola $y = x^2$, to give both a parabolic arch and a bending-moment diagram.

As a further illustration, take the shearing-force diagram just mentioned. We should begin by analysing in even sine harmonics the function defined by

$$y = -x, \quad 0 < x < \tfrac{1}{2}\pi \text{ (fig. 5).}$$

This gives $$y = b_2 \sin 2x + b_4 \sin 4x + \cdots,$$

with $$b_n = -\frac{4}{\pi}\int_0^{\frac{1}{2}\pi} x \sin nx\,dx, \quad n \text{ even},$$

$$= -\frac{4}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\frac{1}{2}\pi}$$

$$= \frac{2}{n}\cos\tfrac{1}{2}n\pi.$$

Thus $$y = -\sin 2x + \tfrac{1}{2}\sin 4x - \tfrac{1}{3}\sin 6x + \dots \quad \text{(i)}$$

Presuming that λ is the length of the beam and w the load per unit length, two changes will reconcile the theoretical figure with the shearing-force diagram. They are (i) the period to be λ instead of 2π, (ii) the terminal ordinate to be $w\lambda/4$ instead of $\frac{1}{2}\pi$. This gives

$$y = \frac{w\lambda}{2\pi}\left\{-\sin\frac{4\pi x}{\lambda} + \tfrac{1}{2}\sin\frac{8\pi x}{\lambda} - \tfrac{1}{3}\sin\frac{12\pi x}{\lambda} + \dots\right\}.$$

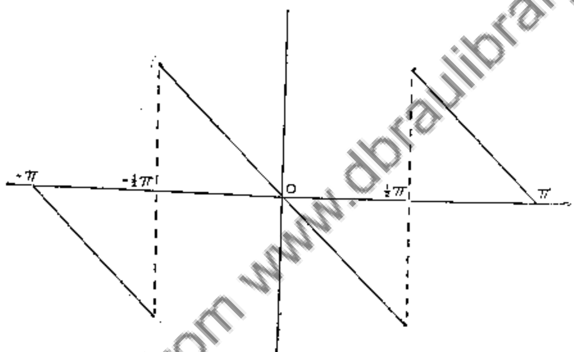

Fig. 5

6, 9. *Series.*

Many interesting series can be obtained from Fourier series by giving particular values to the variable. The results can all be obtained by other means, and they often serve as useful checks on one's working. At points where the function is continuous, the substitution gives the corresponding value of the function; but where the function is discontinuous, the substitution gives the mid-point of the discontinuity.

As a first illustration we take the series 6, 8 (i). The function is continuous at the origin, and the series gives the correct value zero. It is discontinuous at $x = \frac{1}{2}\pi$, and the series gives the mid-point of the discontinuity. It is continuous at $x = \pi/4$, and gives the well-known result

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

The series in 6 (i) represents a function which is continuous everywhere. On giving x the values 0, π, we have

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots,$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.$$

The half-sum of these gives

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$$

It is then easy to show that the value $x = \pi/4$ leads to

$$\frac{\pi^2}{8\sqrt{2}} = \frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + - - + \ldots,$$

and other curious results can be deduced.

6, 10. We append a number of worked examples illustrating various points in the theory and practice.

Example 1.—Consider the function defined by

$$f(x) = x, \quad -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,$$
$$f(x) = -\pi - x, \quad -\pi < x < -\tfrac{1}{2}\pi,$$
$$f(x) = \pi - x, \quad \tfrac{1}{2}\pi < x < \pi.$$

A sketch shows that it is expressible in odd sines and we have

$$b_n = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} x \sin nx\,dx, \; n \text{ odd},$$
$$= \frac{4}{\pi}\left[\frac{\sin nx}{n^2} - \frac{x\cos nx}{n}\right]_0^{\frac{1}{2}\pi}$$
$$= \frac{4}{\pi}\left[\frac{1}{n^2}\sin\tfrac{1}{2}n\pi\right].$$

Hence $$f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x \ldots\right].$$

This example is useful as showing the manner in which the successive terms gradually approximate to the function. The first term is itself quite a good approximation, but it suffers from two defects (see fig. 6). It begins by rising too steeply, with a slope $4/\pi$ instead of unity; and it does not rise high enough, reaching only to a height $4/\pi = 1{\cdot}27$, whereas the vertex of the graph has an ordinate $\frac{1}{2}\pi = 1{\cdot}57$.

The second term $(4/9\pi)\sin 3x$ may be regarded as a first correction. It improves the central height to $40/9\pi = 1{\cdot}42$, but it rather overdoes the correction for initial slope. If

$$y = \frac{4}{\pi}\left(\sin x - \frac{1}{9}\sin 3x\right),$$

then

$$y_0' = \frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos 3x\right)_0$$

$$= \frac{8}{3\pi} = 0{\cdot}85.$$

The second term is negative between 0 and $\pi/3$, or between $2\pi/3$ and π. It is positive between $\pi/3$ and $2\pi/3$, and its general tendency is to correct the first

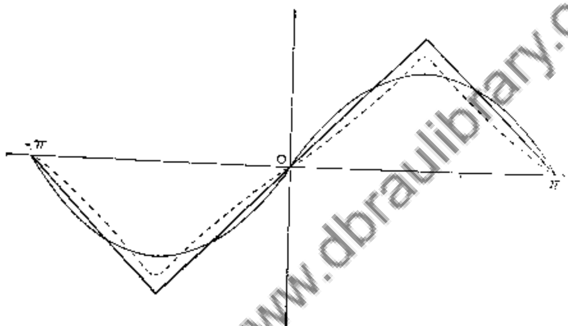

Fig. 6

harmonic nearer to the straight-line graph. The later terms may be regarded as finer additional corrections. Ultimately the initial slope at the origin, as defined by the series, is

$$y_0' = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \dots\right] = 1,$$

and the ordinate at $x = \frac{1}{2}\pi$ is

$$y = \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] = \tfrac{1}{2}\pi.$$

Both these values are correct.

Example 2.—A function $y = f(x)$ is defined by

$$\frac{x}{p} + \frac{y}{q} = 1, \quad 0 < x < 2p,$$

$$\frac{x}{3p} - \frac{y}{3q} = 1, \quad 2p < x < 4p.$$

It is required to express it in a series of period $4p$.

We begin by drawing a rough sketch of the graph, then add a previous period left of the origin. We can now regard a period as covering the range $-2p < x < 2p$. This is evidently symmetrical and expressible in odd cosine harmonics. Its coefficients are therefore to be calculated by taking four times the integral over the quarter period. As the quarter period is p and not $\frac{1}{2}\pi$, we must either alter the rules for determining the coefficients or alter the graph. The latter is preferable and the result can be achieved by a comparison with

$$x + y = \tfrac{1}{2}\pi, \quad 0 < x < \tfrac{1}{2}\pi.$$

This is admittedly in error, both for height and base; but the readjustment can be made after the coefficients have been calculated. We now have

$$\begin{aligned} a_n &= \frac{4}{\pi}\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\pi - x)\cos nx\,dx, \quad n \text{ odd}, \\ &= \frac{4}{\pi}\left[\frac{\pi}{2n}\sin nx - \frac{x}{n}\sin nx - \frac{1}{n^2}\cos nx\right]_0^{\frac{1}{2}\pi} \\ &= \frac{4}{\pi n^2}. \end{aligned}$$

Hence $$y = \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

as the representation of

$$y = \tfrac{1}{2}\pi - x, \quad 0 < x < \tfrac{1}{2}\pi.$$

The vertical height should be q instead of $\frac{1}{2}\pi$, and the period should be $4p$ instead of 2π. Hence

$$y = \frac{8q}{\pi^2}\left[\cos\frac{\pi x}{2p} + \frac{1}{3^2}\cos\frac{3\pi x}{2p} + \frac{1}{5^2}\cos\frac{5\pi x}{2p} + \cdots\right].$$

We can check at various points. Putting $x = 0$ gives $y = q$, whilst $x = p$ gives $y = 0$.

Example 3.—A function is defined by

$$y = 1 - \cos x, \quad 0 < x < \tfrac{1}{2}\pi,$$
$$y = \sin x, \quad \tfrac{1}{2}\pi < x < \pi \text{ (fig 7)}.$$

It is required to express it as a Fourier series of even harmonics.

We have

$$\begin{aligned} \pi a_0 &= \int_0^{\frac{1}{2}\pi}(1 - \cos x)dx + \int_{\frac{1}{2}\pi}^{\pi}\sin x\,dx \\ &= \Big[x - \sin x\Big]_0^{\frac{1}{2}\pi} - \Big[\cos x\Big]_{\frac{1}{2}\pi}^{\pi} = \tfrac{1}{2}\pi, \end{aligned}$$

and $a_0 = \frac{1}{2}$. This can be checked from first principles.

$$\begin{aligned} \tfrac{1}{2}\pi a_2 &= \int_0^{\frac{1}{2}\pi}(1 - \cos x)\cos 2x\,dx + \int_{\frac{1}{2}\pi}^{\pi}\sin x\cos 2x\,dx \\ &= \Big[\tfrac{1}{2}\sin 2x - \tfrac{1}{2}\sin x - \tfrac{1}{6}\sin 3x\Big]_0^{\frac{1}{2}\pi} + \Big[\tfrac{1}{2}\cos x - \tfrac{1}{6}\cos 3x\Big]_{\frac{1}{2}\pi}^{\pi} \\ &= -1. \end{aligned}$$

$$\tfrac{1}{2}\pi b_2 = \int_0^{\frac{1}{2}\pi}(1 - \cos x)\sin 2x\,dx + \int_{\frac{1}{2}\pi}^{\pi}\sin x \sin 2x\,dx$$

$$= \left[-\tfrac{1}{2}\cos 2x + \tfrac{2}{3}\cos^3 x\right]_0^{\frac{1}{2}\pi} + \left[\tfrac{2}{3}\sin^3 x\right]_{\frac{1}{2}\pi}^{\pi}$$

$$= -\tfrac{1}{3}.$$

The higher coefficients can be calculated similarly, and we have

$$y = \tfrac{1}{2} - \frac{2}{\pi}\cos 2x - \frac{2}{3\pi}\sin 2x \ldots .$$

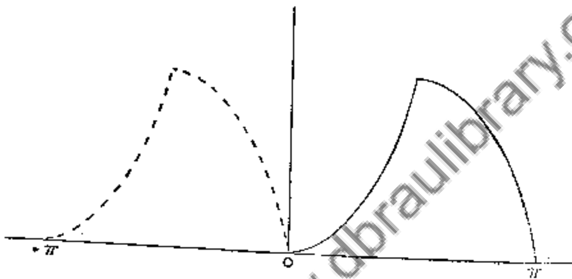

Fig. 7

6, 10·1. *Uniqueness of Period.*

A function of a single variable can have only one fundamental real period. For suppose that α and β are real periods of $f(x)$, then by hypothesis

$$f(x) = f(x + h\alpha) = f(x + k\beta) = f(x + h\alpha + k\beta)$$

for all positive and negative integral values of h and k. If α and β are commensurate, let ω be their greatest common measure, and put $\alpha = m\omega$, $\beta = n\omega$. Then m and n are relatively prime. It is known from the theory of continued fractions that in this case we can find positive or negative integers h, k such that $hm + kn = 1$. Hence

$$f(x) = f(x + h\alpha + k\beta) = f(x + \omega),$$

and ω is a period. Moreover, every other period

$$p\alpha + q\beta = (pm + qn)\omega$$

and must be a multiple of ω, so that in this case there is only one fundamental period.

If, on the contrary, it were possible for α and β to be incommensurate, it is possible to find positive or negative integers h, k such that

$h\alpha + k\beta$ is smaller than any assignable number we like to name in advance. Hence in this case there is no real period, and α, β cannot be incommensurate.

6, 10·2. *The Principle of Least Squares.*

Suppose that the series

$$S(x) = a + bx + cx^2 + \dots$$

is an approximate representation of some function $F(x)$ in a definite range $0 < x < \lambda$. The difference at any point between the value of $F(x)$ and its calculated value $S(x)$ is known as the error. According to the principle of least squares, the sum of the squares of the errors, viz.

$$\int_0^\lambda \{F(x) - S(x)\}^2 dx,$$

should be a minimum. This definite integral is a function of the parameters a, b, c, &c., and if we denote it by $\phi\,(a, b, c, \dots)$ the conditions for a minimum value of ϕ are

$$\frac{\partial \phi}{\partial a} = 0 = \frac{\partial \phi}{\partial b} = \frac{\partial \phi}{\partial c} = \dots .$$

This is the orthodox method of finding the coefficients a, b, c, &c., in any given case.

Applying this to any function $f(x)$ and its Fourier series

$$s(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + \dots$$

we have

$$\int_0^{2\pi} \{f(x) - s(x)\}^2 dx = \int_0^{2\pi} \{f(x)\}^2 dx - 2\int_0^{2\pi} f(x)s(x)\,dx + \int_0^{2\pi} \{s(x)\}^2 dx$$

to be a minimum. If c be indifferently any one of the coefficients, this gives

$$\int_0^{2\pi} f(x) \frac{\partial}{\partial c}\, s(x)\,dx = \int_0^{2\pi} s(x) \frac{\partial}{\partial c}\, s(x)\,dx.$$

If c be interpreted as a_n, we have

$$\int_0^{2\pi} f(x) \cos nx\,dx = \int_0^{2\pi} s(x) \cos nx\,dx = \pi a_n.$$

The conclusion is that the rule for determining the Fourier coefficients agrees with the principle of least squares.

6, 10·3. *The Problem of Dido.*

What is the greatest plane area that can be enclosed by a boundary of given length? Suppose that a simple closed curve of length λ surrounds the origin and crosses OX at A. If P be any point on the curve and s be the length of the arc AP measured counterclockwise, the x, y co-ordinates of P are uniquely determined when s is known. Moreover, they will be periodic functions of s, with period λ, since the addition of λ to s gives the same point P. We may therefore write

$$x = a_0 + \Sigma a_n \cos n\frac{2\pi s}{\lambda} + \Sigma b_n \sin n\frac{2\pi s}{\lambda};$$

$$y = c_0 + \Sigma c_n \cos n\frac{2\pi s}{\lambda} + \Sigma d_n \sin n\frac{2\pi s}{\lambda}.$$

If we choose a new independent variable $\theta = 2\pi s/\lambda$, the range $0 < s < \lambda$ is equivalent to $0 < \theta < 2\pi$ and the co-ordinates become

$$x = a_0 + \Sigma(a_n \cos n\theta + b_n \sin n\theta); \quad y = c_0 + \Sigma(c_n \cos n\theta + d_n \sin n\theta).$$

The fundamental arc-formula

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$$

gives
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{\lambda}{2\pi}\right)^2,$$

and if this last relation be integrated from 0 to 2π, we have

$$\frac{\lambda^2}{2\pi} = \int_0^{2\pi}\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}d\theta = \pi\Sigma n^2(a_n{}^2 + b_n{}^2 + c_n{}^2 + d_n{}^2),$$

$$n = 1, 2, \ldots .$$

This gives the length of the curve in terms of the coefficients. The area of the curve is given by

$$A = \tfrac{1}{2}\int_0^{2\pi}(x\,dy - y\,dx) = \pi\Sigma n(a_n d_n - b_n c_n).$$

We now have

$$\frac{\lambda^2}{2\pi^2} - \frac{2A}{\pi} = \Sigma\{(na_n - d_n)^2 + (nb_n + c_n)^2 + (n^2 - 1)(d_n{}^2 + c_n{}^2)\}.$$

The expression on the right is certainly not negative, so that in general the area A is less than $\lambda^2/4\pi$. We could have $A = \lambda^2/4\pi$ provided

$$na_n - d_n = 0 = nb_n + c_n = (n^2 - 1)(d_n{}^2 + c_n{}^2).$$

For all values of n greater than unity this implies

$$c_n = 0 = d_n = a_n = b_n.$$

When $n = 1$ we have $a_1 = d_1$, $b_1 = -c_1$. Hence the co-ordinates are

$$x = a_0 + a_1 \cos x + b_1 \sin x; \quad y = c_0 - b_1 \cos x + a_1 \sin x,$$

so that the curve is a circle.

The above elegant proof is due to Hurwitz; the problem is usually treated by the calculus of variations. For the classical reference, see Virgil's *Aeneid*, I, 368.

EXERCISES

1. If $f(x) = x$ in the range $-\pi < x < \pi$, prove

$$f(x) = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right).$$

Note that the value $x = \frac{1}{2}\pi$ gives Leibnitz's series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots .$$

2. The symmetric function $f(x) = |x|$, $(-\pi < x < \pi)$ gives

$$f(x) = \tfrac{1}{2}\pi - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right).$$

This is continuous at the origin, and gives

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots .$$

3. If $f(x) = -1$ when $-\pi < x < 0$ and $f(x) = +1$ when $0 < x < \pi$, then

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right).$$

Note that this series can be obtained by differentiating the previous result in No. 2. Differentiation is permissible provided that the resulting series is convergent. Integration is always permissible but may involve the determination of a constant.

4. The expansion $\quad \tfrac{1}{2}x = \dfrac{\sin x}{1} - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \dots$

is valid in the range $-\pi < x < \pi$. Integrate this and determine any constant at the origin. Verify that the result accords with 6, 6 (i).

5. Let $f(x) = x \sin x$ when $-\pi < x < \pi$, then

$$f(x) = 1 - \tfrac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1\,.\,3} - \frac{\cos 3x}{2\,.\,4} + \ldots\right).$$

A check by putting $x = \frac{1}{2}\pi$ gives

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\,.\,3} - \frac{1}{3\,.\,5} + \ldots ,$$

which is a thinly disguised form of Leibnitz's series.

6. The rectified sine curve, $f(x) = |\sin x|$, $-\pi < x < \pi$, gives

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\Sigma\frac{\cos 2nx}{4n^2 - 1}, \quad n = 1, 2, \ldots .$$

Check by putting $x = \frac{1}{2}\pi$.

7. Analyse the function defined by

$$y = x, \quad 0 < x < \pi,$$
$$y = x + \pi, \quad -\pi < x < 0.$$

Verify that the value $x = 0$ gives the middle of the discontinuity.

$[y = \frac{1}{2}\pi - \{\sin 2x + \frac{1}{2}\sin 4x + \frac{1}{3}\sin 6x + \ldots\}.]$

8. A function is defined by $y = x^2$ in the range $0 < x < 1$. Prove that (i) if it is analysed in sines, we have

$$x^2 = \frac{2}{\pi^3}\left[(\pi^2 - 4)\sin \pi x - \tfrac{1}{2}\pi^2 \sin 2\pi x + \left(\frac{\pi^2}{3} - \frac{4}{27}\right)\sin 3\pi x \ldots\right].$$

(ii) If it is analysed in even harmonics, we have

$$x^2 = \tfrac{1}{3} + \frac{1}{\pi^2}\cos 2\pi x + \frac{1}{4\pi^2}\cos 4\pi x + \ldots$$
$$- \frac{1}{\pi}\sin 2\pi x - \frac{1}{2\pi}\sin 4\pi x.$$

Check this last result at the origin.

9. Express $x(\lambda - x)$ in a sine series of period 2λ. Deduce that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots .$$

10. A function is defined as $f(x) = 1 - \cos x$ in the range $0 < x < \frac{1}{2}\pi$. Express it as an odd sine series.

What function does the series represent in the range $\frac{1}{2}\pi < x < \pi$?

$[1 + \cos x.]$

11. From the result 6, 7 (i), deduce

$$\text{(i)}\ \frac{\pi}{4}\tanh\frac{\pi}{2} = \frac{1}{1^2 + 1} + \frac{1}{3^2 + 1} + \frac{1}{5^2 + 1} + \ldots ,$$

$$\text{(ii)}\ \frac{\pi}{4}\operatorname{sech}\frac{\pi}{2} = \frac{1}{1^2 + 1} - \frac{3}{3^2 + 1} + \frac{5}{5^2 + 1} - \ldots .$$

12. If
$$M = (\lambda - c)x \quad \text{when} \quad 0 < x < c,$$
$$M = (\lambda - x)c \quad \text{when} \quad c < x < \lambda,$$
prove that
$$M = \frac{2\lambda^2}{\pi^2}\left\{\sin\frac{\pi c}{\lambda}\sin\frac{\pi x}{\lambda} + \frac{1}{2^2}\sin\frac{2\pi c}{\lambda}\sin\frac{2\pi x}{\lambda} + \dots\right\}.$$

13. The formula for the bending moment at any point of a simply supported beam bearing uniform load w per unit length has been given at the end of 6, 6. Verify that the central bending moment is $w\lambda^2/8$ and zero at each end.

14. It is known (but difficult to prove) that there is only one Fourier series for a function defined in the range $-\pi < x < \pi$. Prove that if the function is defined in only part of the range $-\pi < \alpha < x < \beta < \pi$, it can be represented by an infinite number of Fourier series.

15. Prove that $f(x) + f(-x)$ is symmetrical. What is the analogous skew function?

Deduce that any function can be expressed as the sum of a skew function and a symmetrical function.

If $f(x)$ has a period 2π, write down a related function that is certainly expressible in even harmonics.

16. If m, n are two positive integers, prove that
$$\int_{-\pi}^{\pi}\cos m\theta \cos^n\theta \, d\theta = 0,$$
provided that either (i) m is greater than n, or (ii) m, n are one even, the other odd.

If $m = n$ the value is $\pi/2^{n-1}$.

17. When m is not an integer and $-\pi < x < \pi$, prove that
$$\cos mx = \frac{2m\sin m\pi}{\pi}\left(\frac{1}{2m^2} - \frac{\cos x}{m^2 - 1^2} + \frac{\cos 2x}{m^2 - 2^2} - \frac{\cos 3x}{m^2 - 3^2} + \dots\right).$$
Note that $x = \pi$ gives
$$\pi\cot m\pi = \frac{1}{m} + \Sigma\frac{2m}{m^2 - n^2}, \quad n = 1, 2, \dots .$$
If the last result be integrated with respect to m between 0 and x, we get
$$\sin\pi x = \pi x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\dots ,$$
expressing $\sin\pi x$ as an infinite product. The value $x = \frac{1}{2}$ then gives Wallis's product
$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\dots .$$

18. The speed of a rotor is not constant, but all revolutions take the same time, measured when and where you like. Prove that the angular velocity is independent of the point at which it is measured, and is a periodic function of the time. What is the period?

19. If $y = f(x)$ be expressed in a Fourier series in the range $-\pi < x < \pi$, prove that the mean value of y^2 is greater than the square of the mean value of y.

A rectangular strip of glass is shortened by breakage, the break nowhere doubling back on itself. Taking either part, prove that the distance of the centre of gravity from the clean end is greater than half the mean length.

20. A simply supported beam of length λ has density of loading w per unit length at distance x from the middle of the beam and

$$w = a_0 + a_1 \cos \frac{2\pi x}{\lambda} + \ldots$$

$$+ b_1 \sin \frac{2\pi x}{\lambda} + \ldots .$$

Calculate the reactions and verify that their sum is $a_0\lambda$.

21. An automatic recording instrument gives a polar diagram in the form of a closed curve surrounding the origin. The mean radius is the radius of a circle having the same area. The diagram is re-drawn on squared paper, using equidistant radii as equidistant ordinates. The mean ordinate is the height of a rectangle on the same base and having the same area. Prove that the mean radius is greater than the mean ordinate.

22. Deduce directly from the Fourier series that a symmetric function, defined by $f(x) = f(-x)$, is expressible in cosines only. Proceed similarly for the functions defined by $f(x) = -f(-x)$; $f(x) = f(x + \pi)$; $f(x) = -f(x + \pi)$.

The case of the function defined by

$$f(\tfrac{1}{2}\pi + x) = f(\tfrac{1}{2}\pi - x) = f(-\tfrac{1}{2}\pi - x),$$

in even cosine harmonics, is an exercise in integration.

23. Analyse the function of period 16 defined by

$y = 0,$	$0 < x < 1,$	$7 < x < 9,$	$15 < x < 16,$
$y = 1,$	$1 < x < 3,$	$5 < x < 7,$	
$y = 2,$	$3 < x < 5,$		
$y = -1,$	$9 < x < 11,$	$13 < x < 15,$	
$y = -2,$	$11 < x < 13.$		

24. Analyse the function of period 4 defined by

$y = x + 1,$	$-2 < x < -1,$
$y = -x - 1,$	$-1 < x < 0,$
$y = -x + 1,$	$0 < x < 1,$
$y = x - 1,$	$1 < x < 2.$

CHAPTER VII

Partial Differential Equations

7, 1. It is presumed that the reader is already familiar with the idea of partial differential coefficients and the methods of obtaining them. They make their appearance whenever a measurable quantity is determined by more than one other independent quantity; thus the length of a metal bar may depend on the temperature and the load.

It has already been pointed out that ordinary differential equations can be derived by eliminating the parameters from a family, and the result expresses a property common to the whole family. Similar remarks hold for partial differential equations, except that we are no longer limited to arbitrary parameters but may eliminate arbitrary functions as well. For example, all equations of the form

$$z = f(x) + F(y)$$

can be written as

$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

and the precise forms of f and F are immaterial.

There is one essential difference, which will be pointed out but not proved. In the case of ordinary differential equations the number of parameters in the family determines the order of the eliminant. The analogous statement does not hold in partial differential equations; in fact, in any given case the result of elimination is not even necessarily unique. This need not concern us overmuch, since we do not propose to spend our time eliminating arbitrary functions. The main point is that in solving partial differential equations we must be prepared to see arbitrary functions make their appearance in the solution. The chief difficulty in most problems is in deciding what forms these arbitrary functions shall take.

When the number of variables is three it is an aid to comprehension to view the matter geometrically. We can regard the axis OZ as vertically upward, OY to the right, and OX towards the reader. It can not be too strongly emphasized that if x and y are independent variables, then $\partial y/\partial x$ has no meaning and does not exist. It is sometimes stated quite erroneously, especially in texts on thermodynamics, that it has the value zero. As to which are the independent variables in any particular case, that may be a matter of choice. A unit mass of gas with a characteristic equation has the three coordinates P, V and T, and there are six partial differential coefficients, but $\partial V/\partial T$ exists only on condition that P is an independent variable.

Example.—Consider a surface whose section by any horizontal plane is a circle with its centre on OZ; in other words, a surface of revolution. In general, the radius of this circle will be different at different levels. If in any particular case we knew the connexion between the radius and the height, we could write down the specific equation of the cylinder, cone, sphere, or whatever it was. But if we merely know that there is a connexion between the radius and the height, of the sort that determines the one when the other is known, without being told its specific nature, we write

$$r = f(z), \quad z = F(r).$$

This could just as well be written

$$r^2 = \varphi(z) = x^2 + y^2,$$

or in various other forms, all being the general cartesian equation of surfaces of revolution about OZ. If we regard x, y as the independent variables, we have from the last form on differentiation

$$2x = \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial x}, \quad 2y = \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial y}.$$

Hence
$$x\frac{\partial z}{\partial y} = y\frac{\partial z}{\partial x}.$$

The arbitrary function has disappeared, and we have a partial differential equation in its place.

7, 2. An example of outstanding importance, that well repays any time spent on it, is

$$y = f(x + ct) + F(x - ct). \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

We begin with its interpretation by considering the plane curve $y = f(x)$. If the origin be moved a distance a to the right, the equation becomes $y = f(x + a)$. It is more convenient to regard the axes as fixed and the curve as transported bodily to the left; this gives the same result. If instead of a single movement we have a continuous movement at constant velocity c, the displacement at any time t is ct. Hence the equation $y = f(x + ct)$ represents a curve $y = f(x)$ being carried to the left with constant velocity c. A similar interpretation applies to $F(x - ct)$, moving to the right.

Put $$u = x + ct,\ v = x - ct,$$

so that $$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial x},\quad \frac{\partial u}{\partial t} = c = -\frac{\partial v}{\partial t},$$

and $$y = f(u) + F(v).$$

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial F}{\partial v},$$

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial t} = c\frac{\partial f}{\partial u} - c\frac{\partial F}{\partial v}.$$

Hence $$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 F}{\partial v^2},\quad \frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 F}{\partial v^2}\right),$$

so that $$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}. \qquad \text{(ii)}$$

This is the partial differential equation formed by eliminating the arbitrary functions from (i). Conversely, in lieu of solving this equation (ii) we can say it has the solution (i). As both the functions are completely arbitrary, we can make one of them a constant, or even zero if it suits our purpose. The important point is that this equation (ii) represents phenomena travelling left or right, or both, with constant velocity c. It is known as D'Alembert's equation.

7, 2·1. A particular case of supreme importance occurs when $c^2 = -1$. The interpretation as a travelling phenomenon has now to be dropped; but the formal analysis still holds good, and we have that $f(x + it)$ and $F(x - it)$ are solutions of

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} = 0.$$

This last equation is a two-dimensional form of the more general

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

Known as Laplace's equation, this dominates the whole of classical mathematical physics. We see that its two-dimensional form

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

is satisfied by any function of either of the two conjugate complex variables $x \pm iy$.

7, 3. The importance of the form $f(x + ct)$ lies in its application to travelling phenomena such as electric currents, sound waves, water-hammer in pipes, and so on. When the initial form $f(x)$ is known, the form at any later instant can be written down by merely replacing x by $x - ct$ or $x + ct$, according as the movement is to the right or left. As a simple choice let us suppose that f and F both have the form $A \sin ax$. We then have

$$y = A \sin a(x + ct) + A \sin a(x - ct)$$
$$= 2A \sin ax \cos act.$$

This travels neither to the left nor right. It is a stationary wave $y \propto \sin ax$ of length or period $2\pi/a$, its amplitude $2A \cos act$ being variable with the time. At distances $x = 0, \pm\pi/a, \pm 2\pi/a$, &c., the ordinate y is permanently zero; such points are the nodes. Midway between the nodes the ordinate y periodically varies between $\pm 2A$; such places are known as loops. It is one of the fundamentals of physics that two waves travelling in opposite directions may give stationary waves with nodes and loops.

In the study of any given phenomenon, $f(x)$ is usually in part defined over some finite range with specified end-conditions. The range may be anything from the thickness of a boiler tube to the length of a submarine cable. This robs $f(x)$ of some degree of arbitrariness. Those who are familiar with Fourier series will know that an arbitrary function $f(x)$ over a given finite range can be replaced by a trigonometrical series. The trouble in practice is to choose the right type of series and determine the coefficients.

7, 4. *Longitudinal Waves.*

We can now turn to some of the practical applications, and we begin with longitudinal waves along a bar. In a working pile-driver the monkey is hauled up to a certain height and automatically released to fall on the head of the pile. In accordance with elementary dynamics the pile is immediately driven farther into the ground. It would be nearer the truth to say that the impact of the monkey causes a localized compression which surges down the pile. On the wave reaching the foot of the pile, the shoe is driven farther down. Any suck-back is mainly stopped by the adhesion of the subsoil, and only a small portion of the return compression upsurge reaches the top to give the monkey a slight rebound.

Consider a horizontal bar of length λ to receive a blow at the left end, and suppose that one effect of this is to displace a cross-section of the bar, initially at distance x from some bench mark, to a distance $x + u$. Note that u is not necessarily small; it may be of the same order as x if the bar is free to move.

A section initially at distance $x + dx$ is correspondingly moved to $x + u + dx + du$. This means that the portion of the bar whose length was dx has become $dx + du$. If du is positive there is extension and the stress must be tensile. The strain is $\partial u/\partial x$, and the force over the cross-section is

$$F = Ea\frac{\partial u}{\partial x},$$

where E is Young's modulus and a the cross-sectional area. Similarly, the force at distance $x + dx$ is

$$F + dF = F + \frac{\partial F}{\partial x}dx.$$

The mass of the element between these two sections is $a\rho dx$ if ρ is the density. As the actual movement of the element is u, its velocity is $\partial u/\partial t$, and its acceleration $\partial^2 u/\partial t^2$. This last is caused by the excess of force at one end over the other. Presuming for the sake of argument that the force is tensile, we have F to the left at the left, and $F + dF$ at the right to the right, the balance being dF to the right. On using the ascertained value of F the equation of motion is

$$dF = \frac{\partial}{\partial x}\left(Ea\frac{\partial u}{\partial x}\right)dx = \frac{\partial^2 u}{\partial t^2}a\rho\, dx$$

or

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad \text{(i)}$$

where $c^2 = E/\rho$ and is independent of the sectional area. The result shows that waves of tension, or compression, or both, surge along the bar with velocity $\sqrt{(E/\rho)}$.

The solution of (i) is known to be of the form $f(x + ct)$, or $f(x - ct)$, or a combination of the two. Alternatively, we can employ terms of the type given in 7, 3. The particular form to be adopted depends upon the conditions at the ends of the bar. If an end is fixed, there is no motion there for all time and u must be zero whatever the value of t. If the fixed end is at the left, any form including the factor $\sin nx$ will serve, since this is zero when x is zero, irrespective of the time. If both ends are fixed, a factor $\sin(n\pi x/\lambda)$ will suffice if n is an integer, for this vanishes both for $x = 0$ and for $x = \lambda$.

At an end which is really free, i.e. has no stress applied, the condition is that the stress must be zero. As the stress is tensile or compressive according as $\partial u/\partial x$ is positive or negative, a free end means that $\partial u/\partial x$ is zero. The conditions for a bar fixed at the left and free at the right would be met by a factor $\sin(n\pi x/2\lambda)$ if n is an odd integer; for the differential coefficient with respect to x when $x = \lambda$ is a multiple of $\cos(n\pi/2)$ which is zero for n odd.

Confining our attention to the case of a bar fixed at the left and free at the right, we have as an appropriate solution of (i)

$$y = A \sin \frac{n\pi x}{2\lambda} \cos \frac{cn\pi t}{2\lambda}.$$

The form of the equation (i) shows that if y_1, y_2 are solutions, so likewise is their sum. If one single term does not suffice for our requirements in any particular case, we are at liberty to take a series of them of the necessary type. In the form that we have adopted, the cosine could as well be a sine; the important point is that, as $\sin \omega x$ has a period $2\pi/\omega$, the above sine term has a period or length $4\lambda/n$ where n is odd. If such a sine curve be sketched for simple odd values of n, it will appear that the left end is always a node and the right, or free, end is a loop.

7, 4·1. The present arrangement of the bar, with one end fixed and one end free, found practical application in the Hopkinson pressure-bar for measuring the strength of explosives. Fuller details of the physics of this case and of similar cases need a knowledge of Fourier series, and the treatment can be found in any text on sound.

7, 4·2. It is probably difficult for the reader to resist a feeling of disappointment that the solution is not more cut and dried. He should remember that except in artificial cases the data in this type of problem are somewhat nebulous. When a hammer strikes the head of a chisel there is high local compression; but nobody knows its distribution, and it would pretty certainly be complex anyway. Similarly, a thunder-cloud may induce a high potential as a bound charge on an overhead transmission line. What the distribution is we do not know; what we know, rather regretfully, is that if the charge is unbound by the cloud discharging in a flash of lightning, a high potential will surge right and left along the line. This is our $f(x + ct)$ and $F(x - ct)$.

In some branches of applied mathematics the difficulties of direct attack on problems are so great that inverse methods are adopted. A solution of some fundamental partial differential equation having been obtained, usually by changing the type of co-ordinates employed, we then endeavour to find the problem of which this is the solution.

7, 5. *Transverse Waves.*

Consider a thin flexible wire or inextensible string stretched between two points and vibrating at right angles to its length. It is presumed that the flexural rigidity is negligible, so that the question of shear does not enter into account. Taking the undisturbed position of the string as x axis, let P (with displacement y) be any point in the disturbed position at distance x from the left end, T the tension at P, and ψ the slope of the tangent. The horizontal component at P is $X = T\cos\psi$. At an adjacent point Q, at distance $x + dx$, the corresponding component is $X + dX$, so that the resultant is dX to the right. Presuming that the element PQ has no motion in the direction of its length, we have $dX = 0$, or

$$X = T\cos\psi = \text{const.}$$

The angle ψ is very small, even when the vibrations are visible, so that without serious error we may write

$$\psi = \sin\psi = \tan\psi = \frac{\partial y}{\partial s} = \frac{\partial y}{\partial x}.$$

As $\cos\psi$ is practically unity, we have T constant. This is the usual assumption, and the effect of gravity is ignored.

The inward, or restoring, component at P is $Y = T\sin\psi$, so that at Q it is outward and of magnitude $Y + dY$. The resultant is dY outward, and this causes the acceleration $\partial^2 y/\partial t^2$. The mass of the element PQ is $m\,dx$, if m is the mass per unit length, and its equation of motion is

$$dY = \frac{\partial}{\partial x}(T\sin\psi)\,dx = T\frac{\partial^2 y}{\partial x^2}\,dx = \frac{\partial^2 y}{\partial t^2}\,m\,dx,$$

or

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}, \quad c^2 = \frac{T}{m}.$$

If the string has its ends fixed at distance λ apart, we have

$$y = A\sin\frac{n\pi x}{\lambda}\cos\frac{cn\pi t}{\lambda}$$

as a suitable solution when n is an integer. This makes y always zero if x is λ or zero.

In the simplest case, $n = 1$, and

$$y = A \sin \frac{\pi x}{\lambda} \cos \frac{c\pi t}{\lambda}.$$

The string always has the form of half a sine wave of period 2λ. The amplitude $A \cos(c\pi t/\lambda)$ varies with the time; this, of course, constitutes the vibration. The number of vibrations per second is $c/2\lambda$ and is known as the fundamental. It increases with increasing T, but decreases with increasing λ or m. A glance inside a piano will illustrate the point. If $n = 2$, the string has the form of a complete sine wave of length λ and the frequency is now c/λ, or doubled. There is an intermediate node, and the tone is known as the first harmonic; or, musically, the octave. The higher harmonics correspond to higher integer values of n.

7, 5·1. It is an extremely difficult matter to produce a pure tone, and recent research shows that the human ear has a knack of supplying the harmonics, even when they are not produced. All instruments produce harmonics with their fundamental, and the combination gives them their characteristic quality, or timbre, that enables us to distinguish one from another. Readers interested in the matter should read Jeans' *Science and Music*. The stretched wire has found practical application in the Maihak extensometer.

7, 6. The solution of our last two problems has been not so much a matter of solving a partial differential equation as of applying a known solution; we have been trading on the fact that $f(x + ct)$ and $f(x - ct)$ are solutions of

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

When we get away from an equation so simple as this it is desirable to have a general method that covers most of the commonly occurring cases.

If a variable V depends on two independent variables x and y, the standard method is to substitute $V = XY$, where X is a function of x alone, and Y of y alone. The hope is that in the resulting equation we can separate the terms containing x from those containing y. An equation of the form $f(x) + F(y) = 0$, where x and y are quite independent variables, can hold only if $f(x) = \text{const.} = -F(y)$. The above substitution may thus lead to two ordinary differential equations for the determination of X and Y.

Example 1.—Consider Laplace's equation in two dimensions,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

The substitution $V = XY$ gives

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0,$$

whence $$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}.$$

If we equate each of these to the constant n^2, we have the two equations

$$\frac{d^2X}{dx^2} - n^2X = 0, \quad \frac{d^2Y}{dy^2} + n^2Y = 0.$$

These have the solutions $\quad X = e^{nx}, \quad Y = e^{iny},$

so that $$V = e^{n(x+iy)}.$$

Alternatively, $\quad X = \sinh nx, \ \cosh nx; \quad Y = \sin ny, \ \cos ny,$

so that $V = \sin ny \cosh nx$, or any linear combination of similar forms with arbitrary coefficients.

Example 2.—Consider similarly the equation

$$\frac{\partial V}{\partial t} = m\frac{\partial^2 V}{\partial x^2}.$$

The procedure $V = XT$ gives

$$X\frac{\partial T}{\partial t} = mT\frac{\partial^2 X}{\partial x^2},$$

or $$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \text{const.} = \frac{1}{mT}\frac{\partial T}{\partial t}.$$

There are no limitations on this constant; it need not be real, so we examine the result of choosing a complex constant $(a + ib)^2$. This gives the two equations

$$\frac{d^2X}{dx^2} = (a+ib)^2X; \quad \frac{dT}{dt} = mT\{(a^2 - b^2) + 2iab\}.$$

The former is satisfied by $D = \pm(a + ib)$, so that a solution could be

$$X = e^{-ax}e^{-ibx}.$$

The latter is satisfied by

$$T = \exp\{mt(a^2 - b^2)\}\exp(2imabt).$$

Jointly, we have

$$V = XT = e^{-ax}e^{mt(a^2-b^2)}\exp ib\{2mat - x\}.$$

As this has a real and an imaginary part, we might have

$$V = e^{-ax}e^{mt(a^2-b^2)}\sin b(2mat - x).$$

The sine shows this to be some sort of wave travelling with velocity $2ma$. It is damped in space by the exponential factor e^{-ax}. Whether it grows or decays with time depends on whether a is greater or less than b, presuming that a is positive.

The variety of choice in the arbitrary constant opens up wide possibilities. If we choose a negative value $-c^2$, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -c^2 = \frac{1}{mT}\frac{dT}{dt}.$$

This gives $X = \sin cx, \cos cx$; $T = e^{-mc^2t}$ and an admissible value of V is

$$V = Ae^{-mc^2t}\cos cx.$$

This represents a stationary wave distribution damped in time. We get an interesting corollary if we take a composite solution formed by giving c the successive integer values. We have

$$V = A_1e^{-mt}\cos x + A_2e^{-4mt}\cos 2x + A_3e^{-9mt}\cos 3x + \ldots .$$

The initial value of V, obtained by equating t to zero, is

$$V = A_1\cos x + A_2\cos 2x + A_3\cos 3x + \ldots .$$

The wave lengths of these components successively decrease; but their negative exponential damping factors decrease far more rapidly. The short-wave components are the most heavily damped.

We might even take a zero constant, so that

$$\frac{1}{X}\frac{d^2X}{dx^2} = 0 = \frac{1}{mT}\frac{dT}{dt}.$$

These give T constant and $X = a + bx$, so that $V = A + Bx$. This result, being invariable with time, is known as a "steady state". In the present instance it is a straight-line distribution.

The conclusion to be drawn from the foregoing is that the solution of a partial differential equation is usually a very flexible affair; with a little ingenuity it can be made to cover a wide variety of cases. In practice one usually has a pointer to the sort of solution required, if only in the fact that effects usually die off at a distance.

7, 7. *Heat Conduction.*

Consider the conduction of heat along a uniform unlagged bar. Take a section P at distance x from the left and let Q at $x + dx$ be an adjacent section. If heat H crosses the section P from the left, then $H + dH$ similarly crosses Q. Suppose E is the amount emitted by the surface of the element PQ, and let R be the addition of heat to the element itself, shown by a rise of temperature. We then have

$$H = E + R + H + dH,$$

or

$$E + R = -\frac{\partial H}{\partial x}dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where the quantities can be taken to apply per second. We have

$$H = -as\frac{\partial \theta}{\partial x},$$

where a is the cross-sectional area, s the conductivity, and θ the temperature at distance x from the left. Also $E = \theta hp\,dx$, where h is the emissivity and p the perimeter, so that $p\,dx$ is the surface area. The access of heat is

$$R = k\frac{\partial \theta}{\partial t}\rho a\,dx,$$

where ρ is the density and k the specific heat. The equation (i) becomes

$$\theta hp\,dx + k\frac{\partial \theta}{\partial t}\rho a\,dx = as\frac{\partial^2 \theta}{\partial x^2}dx,$$

or
$$\frac{\partial^2 \theta}{\partial x^2} = A\frac{\partial \theta}{\partial t} + B\theta, \quad \ldots \ldots \quad \text{(ii)}$$

where
$$A = \frac{\rho k}{s}; \quad B = \frac{hp}{as}.$$

This equation is slightly less simple than our last example; but no more difficult to solve. Note that if the bar is lagged to prevent radiation, $h = 0 = B$, and we reach the form previously given in 7, 6. Alternatively, if the steady state has been achieved, so that $\partial\theta/\partial t = 0$, the equation (ii) ceases to be partial and takes a form already discussed in 3, 4·2.

The substitution $\theta = XT$ leads to

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{A}{T}\frac{dT}{dt} + B = \text{const.}$$

If, as is usual, we desire a solution that is trigonometrical in x, we take the constant to be $-n^2$. This gives

$$X = P\cos nx + Q\sin nx; \quad T = e^{-ct},$$

where
$$c = (n^2 + B)/A.$$

Hence
$$\theta = e^{-ct}(P\cos nx + Q\sin nx),$$

where P, Q, n are arbitrary. The initial value, given by $t = 0$, is

$$\theta = R\sin(nx + \phi),$$

or the sum of any number of similar terms.

7, 8. *The Transverse Vibrations of a Rod.*

The vibrations of a rod, as distinct from a wire, are mainly due to the rod's flexural rigidity. We consider an element contained between a section at A, distance x from the left, and a neighbouring section at B, at distance $x + dx$. Taking couples as counterclockwise positive, and y to be the lateral displacement of the neutral axis at A, the bending moment M at A is

$$-M = -EI \frac{\partial^2 y}{\partial x^2}.$$

The corresponding couple at B is $M + dM$, so that there is a resultant dM counterclockwise. This is counterbalanced by a couple due to the two shears. Presuming that the shear is Q outwards at A, it will be $Q + dQ$ inwards at B, and we have $dM = Q\,dx$. There is now a resultant force dQ inwards (or $-dQ$ outwards), and this causes the acceleration $\partial^2 y/\partial t^2$. Taking m as the mass per unit length, we have, neglecting gravity,

$$m\,dx \frac{\partial^2 y}{\partial t^2} = -dQ = -\frac{\partial^2 M}{\partial x^2} dx = -EI \frac{\partial^4 y}{\partial x^4} dx,$$

or

$$\frac{\partial^4 y}{\partial x^4} + \frac{m}{EI} \frac{\partial^2 y}{\partial t^2} = 0.$$

As a departure from the routine procedure, and knowing that the rod vibrates, we might put

$$y = X \sin(pt + \phi)$$

and hope to determine the possible values of p. This substitution gives

$$\frac{d^4 X}{dx^4} = a^4 X; \quad a^4 = \frac{p^2 m}{EI}$$

and

$$X = A \cos ax + B \sin ax + C \cosh ax + D \sinh ax$$

(see 3, 6, Exercise 7). The three ratios of the four coefficients, together with a or p, give four unknowns. They are determined from the four terminal conditions, two at each end.

As all the possible cases are treated similarly, let us suppose the left end clamped and the right end merely held. Taking the origin at the left, we have

$$x = 0 = y = \frac{\partial y}{\partial x},$$

so that
$$X = 0 = \frac{dX}{dx},$$

and
$$A + C = 0 = B + D.$$

Hence $$X = A(\cos ax - \cosh ax) + B(\sin ax - \sinh ax).$$

If λ be the length of the bar, we have at the right $X = 0$ when $x = \lambda$,

so that $$0 = A(\cos a\lambda - \cosh a\lambda) + B(\sin a\lambda - \sinh a\lambda).$$

A further condition is that no couple is applied at the right, so that
$$\frac{\partial^2 y}{\partial x^2} = 0 = \frac{d^2X}{dx^2}.$$

Hence $$0 = -Aa^2(\cos a\lambda + \cosh a\lambda) - Ba^2(\sin a\lambda + \sinh a\lambda).$$

On dividing by a^2 we have on addition and subtration
$$A \cos a\lambda + B \sin a\lambda = 0,$$
$$A \cosh a\lambda + B \sinh a\lambda = 0.$$

The elimination of the ratio A/B gives $\tan a\lambda = \tanh a\lambda$, so that our answer depends on the solution of the transcendental equation
$$\tan x = \tanh x.$$

The graphs of both of these functions pass through the origin at 45° and a rough sketch shows that the first root is slightly short of $5\pi/4$. The higher roots are given approximately by the successive addition of π, so that
$$x = a\lambda = \frac{5\pi}{4}, \quad \frac{9\pi}{4}, \; \ldots .$$

The frequency of the slowest mode of vibration is $p/2\pi$, where
$$p = a^2\left(\frac{EI}{m}\right)^{\frac{1}{2}} = \left(\frac{5\pi}{4\lambda}\right)^2\left(\frac{EI}{m}\right)^{\frac{1}{2}}.$$

Comparing the result with the corresponding result for a vibrating string, we see that the simple relation between the fundamental and the various harmonics no longer holds.

7, 9. *Whirling of Shafts.*

The foregoing discussion is closely allied to a phenomenon known as the whirling of shafts. It is a matter of experience that long rotors at certain speeds, known as critical speeds, exhibit a species of instability accompanied by considerable vibration. Their length departs from the dead straight line, and to put it crudely for the sake of emphasis they behave rather like a skipping rope. Judging by the fact that scientific papers on the subject continue to be published, the dynamics of the problem is still a matter for discussion; but the existence of the phenomenon is not in dispute. These critical speeds are not necessarily unique; there may be more than one of them, and, in fact, a high-speed rotor like a steam turbine is habitually operated above its first critical speed.

We can adopt a simplified treatment by assuming that our element AB is displaced a distance y from the original centre line and rotates with angular velocity ω in a circle of radius y. Since its mass is $m\,dx$, a force directed to the centre and of magnitude $m\omega^2 y\,dx$ is required to maintain the circular motion. This is the so-called centrifugal force, and it is supplied by the element of shear dQ. Hence

$$m\omega^2 y\,dx = dQ = \frac{d^2M}{dx^2}dx = EI\,\frac{d^4y}{dx^4}\,dx,$$

or

$$\frac{d^4y}{dx^4} = \alpha^4 y; \quad \alpha^4 = \frac{m\omega^2}{EI}.$$

The solution has already been given. The values of α, and hence the values of ω, will be determined by the mode of fixing the shaft. Comparing our present α with our previous α in 7, 8, we see that for a given mode of support they are determined in the same manner, and hence for a specified shaft they must have the same values. For example, if λ is the length and both ends are in long bearings, we have $y = 0 = y'$ both when $x = 0$ and when $x = \lambda$. Hence the solution depends on $\cos\alpha\lambda = \cosh\alpha\lambda$ or on $\cos\alpha\lambda = \cosh\alpha\lambda$ as the case may be; see the reference to 3, 6, given in 7, 8. It follows that ω and p will be the same and $\omega/2\pi = p/2\pi$. In other words, the rotational frequency when whirling is the free frequency when transversely vibrating. This is strong evidence in favour of the belief that whirling is closely allied to resonance.

7, 10. It occasionally happens that a partial differential equation is the eliminant from two or more partial simultaneous equations. We consider as an illustration the case of a leaky transmission line. Let R be the resistance, C the capacity and L the inductance, all per unit length. Let G be the leakance; this is defined for unit voltage per unit length. Consider two sections A and B at distances x and $x + dx$ from the left: presume that the voltage is V at A and I is the current to the right. At B the corresponding quantities will be $V+dV$ and $I+dI$, so that the voltage drop from A to B is $-dV$. The quantities relating to our element AB are dR, dC and dL. Ohm's law gives

$$I\,dR = -dV - \frac{\partial I}{\partial t}\,dL.$$

Whatever current passes A and fails to pass B must either leak away or charge-up the capacity. Hence per second

$$I = VdG + \frac{\partial V}{\partial t}\,dC + I + dI.$$

Since $dR = Rdx$, $dV = \frac{\partial V}{\partial x}\,dx$, &c., we have the simultaneous equations

$$-\frac{\partial V}{\partial x} = RI + L\frac{\partial I}{\partial t} = \left(R + L\frac{\partial}{\partial t}\right)I. \quad . \quad . \quad . \quad \text{(i)}$$

$$-\frac{\partial I}{\partial x} = VG + C\frac{\partial V}{\partial t} = \left(G + C\frac{\partial}{\partial t}\right)V. \quad . \quad . \quad \text{(ii)}$$

We can eliminate I by operating on (i) with $\partial/\partial x$ and on (ii) with $\left(R + L\frac{\partial}{\partial t}\right)$. The result is

$$CL\frac{\partial^2 V}{\partial t^2} + (CR + GL)\frac{\partial V}{\partial t} + GRV = \frac{\partial^2 V}{\partial x^2}. \quad . \quad . \quad \text{(iii)}$$

Certain variants of this equation are known by special names. If R and G are small, or the frequency high, the dominant elements are C and L. The equation then has the form

$$CL\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 V}{\partial x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

This form, known as the "radio equation", has already been discussed. It is known to represent waves travelling with the velocity $1/\sqrt{(CL)}$, the same for all wave lengths.

In telegraph work the variations in I or V are small enough to permit the ignoration of L. The cables are well insulated, and G is small. The equation thus reduces to

$$\frac{\partial^2 V}{\partial x^2} = CR\frac{\partial V}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

and is known as the "telegraph equation". The same form applies to the conduction of heat along a lagged bar, and has already been discussed.

A leaky telegraph wire is characterized by negligible capacity and inductance. In this case the equation ceases to be partial and becomes the ordinary equation

$$\frac{d^2V}{dx^2} = GRV.$$

This can be solved in hyperbolic functions; the quantity $\alpha = \sqrt{(GR)}$ is known as the "attenuation constant".

Reverting to the equation (iii), the possible solutions are certain to be very varied; but since there is a certain amount of leakance, the transmissions will weaken at a distance. We might therefore legitimately assume $V = e^{-\alpha x}U$. This gives

$$\frac{\partial^2 V}{\partial x^2} = e^{-\alpha x}\left\{\frac{\partial^2 U}{\partial x^2} - 2\alpha\frac{\partial U}{\partial x} + \alpha^2 U\right\}$$

$$= e^{-\alpha x}\left\{CL\frac{\partial^2 U}{\partial t^2} + (CR + GL)\frac{\partial U}{\partial t} + RGU\right\}.$$

At first sight we appear to have made confusion worse confounded; but as the transmissions are to travel along the line we might tentatively put $U = \sin\beta(x - vt)$. This is a sine wave of length $2\pi/\beta$ travelling with velocity v. On dropping the exponential factor and substituting we have

$$(\alpha^2 - \beta^2)\sin\theta - 2\alpha\beta\cos\theta = (RG - CLv^2\beta^2)\sin\theta - \beta v(RC + GL)\cos\theta,$$

where θ has been written for $\beta(x - vt)$. If this relation is to hold for all values of x and t, the corresponding coefficients must be equal, and we have

$$\alpha^2 - \beta^2 = RG - CLv^2\beta^2, \quad 2\alpha = v(RC + GL).$$

The results show that the velocity v is dependent on the damping α, whilst β and the wave length are dependent on v and α jointly. Accordingly the waves travel at various speeds, dependent on their wave length, and are variously affected by the damping. In practice this is highly objectionable. A transmitted signal has numerous sine and cosine components, and if these travel at different speeds the signal may arrive at the receiving end distorted beyond recognition. It will be observed, however, that if $\alpha^2 = RG$ the velocity is uniquely given as $v^2 = 1/CL$. In this case $4\alpha^2 = v^2(RC + GL)^2$, whence

$$4RG = (RC + GL)^2/CL.$$

This is equivalent to $(RC - GL)^2 = 0$, or $RC = GL$. With these conditions satisfied, all wave lengths travel at the same speed and suffer like attentuation. This state of affairs is so desirable that special steps are taken to ensure it. Details will be found in any text dealing with Pupin's theory of the loaded cable.

7, **11.** We append a number of worked examples illustrative of the types of difficulties encountered.

Example 1.—In a telegraph cable of length λ where the voltage is governed by the equation 7, 10 (v), the receiving end is earthed and the transmitting end has a constant applied voltage V_0. When a steady state has been reached the transmitting end is suddenly earthed. It is required to find the voltage distribution at any subsequent time.

In the steady state, $\partial V/\partial t = 0$, so that $\partial^2 V/dx^2 = 0$, and we have the straight-line distribution $V = a + bx$. As end conditions for the determination of these arbitrary constants, we have $V = 0$ when $x = \lambda$, and $V = V_0$ when $x = 0$. Hence

$$V = V_0 \frac{\lambda - x}{\lambda}. \quad \text{(i)}$$

This result is obvious from a sketch.

Subsequently the voltage is to be zero at both ends. This suggests that we seek a solution of 7, 10 (v) in terms of $\sin(n\pi x/\lambda)$, where n is an integer; for this has the property of being zero both when $x = 0$ and when $x = \lambda$, irrespective of the time. The substitution

$$V = T \sin \frac{n\pi x}{\lambda}$$

gives

$$-\frac{n^2\pi^2}{\lambda^2} T = CR \frac{dT}{dt},$$

so that

$$T = \exp(-n^2\alpha t), \quad \alpha = \frac{\pi^2}{CR\lambda^2}.$$

Hence we have the tentative solution

$$V = \Sigma B_n e^{-n^2\alpha t} \sin \frac{n\pi x}{\lambda}, \quad \text{(ii)}$$

valid at time t. The initial value of this, when $t = 0$, is

$$V = \Sigma B_n \sin \frac{n\pi x}{\lambda}. \quad \text{(iii)}$$

The problem now is to make the voltage distribution (iii) coincide with (i), with the additional fact that $V = 0$ when $x = 0$ owing to the earthing of the transmitting end. We have recourse to Fourier series, and consider the function defined by

$$y = \pi - x, \quad 0 < x < \pi.$$

It is left to the reader to verify that if this be expressed in sines over a range 2π, we have

$$y = 2[\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \ldots].$$

Two modifications are now required. They are, firstly, the theoretical half-range π is to be changed to λ; secondly, the peak-height is to be changed from π to V_0. This gives

$$y = \frac{2}{\pi} V_0\left[\sin\frac{\pi x}{\lambda} + \tfrac{1}{2}\sin\frac{2\pi x}{\lambda} + \tfrac{1}{3}\sin\frac{3\pi x}{\lambda} + \ldots\right]$$

as the voltage distribution at the instant of earthing the transmission end. In accordance with (ii) the voltage distribution at any subsequent time t is

$$V = \frac{2}{\pi} V_0\left[e^{-at}\sin\frac{\pi x}{\lambda} + \tfrac{1}{2}e^{-4at}\sin\frac{2\pi x}{\lambda} + \tfrac{1}{3}e^{-9at}\sin\frac{3\pi x}{\lambda} + \ldots\right].$$

Example 2.—It is required to find a solution of the equation

$$r^2\frac{d^2w}{\partial r^2} + r\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial\theta^2} = ar^2$$

such that w shall vanish on the periphery of an ellipse.

It will be observed that there is a particular integral

$$w = h + k\theta + cr^2$$

provided $4c = a$. Hence if we put $w = u + h + k\theta + \tfrac{1}{4}ar^2$, we are left to solve the equation

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial\theta^2} = 0.$$

This can be solved by the general method of 7, 6, and if we take $u = R\cos n\theta$ or $R\sin n\theta$, we have the homogeneous equation

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0.$$

Employing the method of 4, 5, we have $R = r^n$ or r^{-n}.

Since an ellipse is of the second order we can take $n = 2$, and we have as a likely solution

$$w = Ar^2\cos 2\theta + Br^2\sin 2\theta + h + \tfrac{1}{4}ar^2.$$

On replacing the trigonometrical terms by cartesians, we have

$$w = x^2(A + \tfrac{1}{4}a) + 2Bxy + y^2(\tfrac{1}{4}a - A) + h.$$

This certainly makes w vanish on the periphery of an ellipse if we make B zero, give h a negative value, and take A to be positive and less than $\tfrac{1}{4}a$. The level lines for w are then similar and similarly situated ellipses. It is evidently not essential that B should be zero; there are other suitable values that would fulfil the conditions of the problem.

Example 3.—To find the partial differential equation of all developable surfaces.

A developable surface may be defined as the envelope of a plane whose equation contains a single parameter. We can take as the variable plane

$$ax + by + cz + d = 0,$$

where each of the coefficients is a function of some parameter β. On differentiating separately with respect to x and y, we have

$$a + c\frac{\partial z}{\partial x} = 0 = b + c\frac{\partial z}{\partial y}.$$

As each of the fractions a/c and b/c is a function of β, the one is a function of the other and we have

$$\frac{\partial z}{\partial x} = f\left(\frac{\partial z}{\partial y}\right).$$

To eliminate the arbitrary function f we again differentiate separately with respect to x and y. This gives

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial x \partial y} f'\left(\frac{\partial z}{\partial y}\right), \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y^2} f'\left(\frac{\partial z}{\partial y}\right).$$

On eliminating f' by cross-multiplication, we have

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

as the equation of all developable surfaces. It is customary to use the notation

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t,$$

so that the result is usually written $rt = s^2$.

Example 4.—A uniform plate of thickness λ has a temperature distribution which is uniform over planes parallel to its faces. It cools by radiation into a medium at temperature zero. Discuss the temperature changes, neglecting any effects at the edges.

This seemingly innocent problem opens up a new field of difficulties. Take a normal to the faces as x axis and the origin at the middle of the plate. There is a one-dimensional heat-transfer, so that from 7, 7 (ii) we can take

$$\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2}.$$

The rate of radiation depends on the conductivity, emissivity and the negative temperature gradient. Hence

$$h\theta = -\frac{\partial \theta}{\partial x} \quad \text{when} \quad x = \tfrac{1}{2}\lambda,$$

where h is some constant. Similarly,

$$h\theta = +\frac{\partial \theta}{\partial x} \quad \text{when} \quad x = -\tfrac{1}{2}\lambda.$$

As a tentative solution we can take

$$\theta = e^{-kc^2 t}(A\cos cx + B\sin cx),$$

so that

$$\frac{\partial \theta}{\partial x} = e^{-kc^2 t}(cB\cos cx - cA\sin cx).$$

So far c is arbitrary, but it will appear later that its possible values are dictated by the surface conditions. The boundary condition on the right gives

$$h(A\cos\tfrac{1}{2}c\lambda + B\sin\tfrac{1}{2}c\lambda) = c(A\sin\tfrac{1}{2}c\lambda - B\cos\tfrac{1}{2}c\lambda).$$

Similarly at the left we have

$$h(A\cos\tfrac{1}{2}c\lambda - B\sin\tfrac{1}{2}c\lambda) = c(A\sin\tfrac{1}{2}c\lambda + B\cos\tfrac{1}{2}c\lambda).$$

Hence
$$A(h\cos\tfrac{1}{2}c\lambda - c\sin\tfrac{1}{2}c\lambda) + B(c\cos\tfrac{1}{2}c\lambda + h\sin\tfrac{1}{2}c\lambda) = 0,$$
$$A(h\cos\tfrac{1}{2}c\lambda - c\sin\tfrac{1}{2}c\lambda) - B(c\cos\tfrac{1}{2}c\lambda + h\sin\tfrac{1}{2}c\lambda) = 0.$$

If we give B the value zero, we assume that the temperature distribution is symmetrical about the middle of the plate. Similarly, to make A zero is to posit a skew distribution. In general, therefore, neither A nor B will be zero. This necessitates both

$$h\cos\tfrac{1}{2}c\lambda - c\sin\tfrac{1}{2}c\lambda = 0,$$

and
$$c\cos\tfrac{1}{2}c\lambda + h\sin\tfrac{1}{2}c\lambda = 0,$$

so that
$$\tan\tfrac{1}{2}c\lambda = \frac{h}{c} \quad\text{and}\quad -\frac{c}{h}.$$

The two results can be reconciled by doubling the angle, which gives the unique result

$$\tan c\lambda = \frac{2hc}{c^2 - h^2}$$

as the equation for determining c. It is more conveniently written in the form

$$\cot c\lambda = \frac{c}{2h} - \frac{h}{2c}.$$

By sketching the graphs of the two sides, the right side being a hyperbola, we see that there are positive roots between 0 and $\frac{1}{2}\pi$, π and $3\pi/2$, &c., but otherwise the roots are irregularly disposed. If we call the roots c_1, c_2, &c., we have the solution

$$\theta = \Sigma(A_r\cos c_r x + B_r\sin c_r x)\exp(-kc_r^2 t).$$

The initial temperature distribution, given by $t = 0$, is

$$\theta_0 = \Sigma(A_r\cos c_r x + B_r\sin c_r x).$$

Whether this can be made to coincide with any arbitrary distribution $f(x)$ is somewhat analogous to a problem in Fourier series, with the very considerable difference that the c's no longer have integral ratios. The matter is discussed further in texts on heat conduction, e.g. Carslaw, *The Conduction of Heat.*

EXERCISES

1. Verify that the alternative forms of the surface of revolution, viz. (i) $r = f(z)$; (ii) $z = F(x^2 + y^2)$, lead to the partial differential equation given in the text 7, 1.

2. A moving horizontal line intersects OZ and rotates as it rises. If its plan makes θ with OX when its height is z, we have $z = f(\theta)$ or $\theta = F(z)$. As $\tan\theta = y/x$, we can write the family as $z = \varphi(\tan\theta)$, or $y/x = \psi(z)$, and so on. The surface is

a helicoid, like a screw of variable pitch. In the language of geometry it is ruled but not developable. Prove that, whatever form we take for the cartesian equation, the partial differential equation is

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0,$$

and compare the result with Euler's theorem on homogeneous functions.

3. A moving straight line always passes through the origin, and hence describes some sort of cone. Its direction cosines are x/r, y/r and z/r, where $r^2 = x^2 + y^2 + z^2$, and a surface could be defined by a relation between any two of the direction cosines. Moreover, if the line be inclined at φ to OZ, and if its plan be inclined at θ to OX, a relation between θ and φ would define a surface. Verify that in any case the cartesian equation of the conical surface satisfies the equation

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y},$$

and comes in the class of developable surfaces.

4. Prove that $y = Ae^{-\alpha t}\sin\beta x$ satisfies $\dfrac{\partial^2 y}{\partial x^2} = c\dfrac{\partial y}{\partial t}$, where $c = \beta^2/\alpha$.

5. Verify by substitution that any term of the form

$$y = A\,\frac{\sin}{\cos}\,nx\,\frac{\sin}{\cos}\,cnt$$

is a solution of
$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}.$$

6. By means of the substitution $u = x + iy$, $v = x - iy$, and the fact that

$$\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\theta}{\partial v}\frac{\partial v}{\partial x},$$

show that the equation
$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

transforms to
$$\frac{\partial^2\theta}{\partial u\,\partial v} = 0,$$

and therefore has the solution $\theta = f(u) + F(v)$.

7. Prove that $a\dfrac{\partial z}{\partial x} + b\dfrac{\partial z}{\partial y} = 0$ has the solution $z = f(bx - ay)$. Hence obtain solutions of

$$\text{(i)}\quad a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + c = 0.$$

$$\text{(ii)}\quad a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cx = 0.$$

$$\text{(iii)}\quad a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0.$$

8. Prove that $z = e^{ax}\sin ax \sin \frac{2a^2}{c} y$ is a solution of

$$\frac{\partial^4 z}{\partial x^4} = c^2 \frac{\partial^2 z}{\partial y^2}$$

for all values of a, and deduce other similar solutions, including those involving hyperbolic functions.

9. Prove that Laplace's equation in two dimensions (see 7, 2·1) is separately satisfied by (i) the real part, (ii) the imaginary part of $f(x+iy)$. Deduce that $\log(x^2+y^2)$ is a solution.

What linear combinations of x, y, x^2, y^2, xy are solutions?

10. Find the most general non-homogeneous quadratic in x and y which is a particular integral of

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = k.$$

11. Find cubic polynomial solutions of Laplace's equation in two dimensions, and show that

$$V = A(x-y)(x^2+4xy+y^2)$$

is a solution such that the equipotential curves $V = \text{constant}$ are orthogonal to the three straight lines

$$(x+y)(x^2-4xy+y^2) = 0.$$

12. It is required to find solutions of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + p^2 V = 0,$$

which shall make V zero when $x = \pm a$, $y = \pm b$. Prove that

$$V = A \begin{matrix}\sin\\ \cos\end{matrix} \frac{n\pi x}{a} \begin{matrix}\sin\\ \cos\end{matrix} \frac{m\pi y}{b}$$

is suitable provided certain relations hold among the constants. Deduce that if $a = b$ their smallest permissible value is $\pi/p\sqrt{2}$.

13. The general method of 7, 6 necessarily fails when applied to equations of the form

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x\,\partial y} + c\frac{\partial^2 z}{\partial y^2} = 0.$$

Obtain solutions by the assumption $z = e^{\beta x} Y$, and examine the possible cases.

14. Deduce the radio equation 7, 10 (iv) from first principles; likewise the telegraph equation 7, 10 (v).

15. In a uniform bar of length λ the cross-section at distance x from one end is turned as a whole through angle θ. Prove that the torque at the section is proportional to $\partial\theta/\partial x$, and deduce that waves of torsion are transmitted along the bar in accordance with an equation of D'Alembert's type.

What are the conditions for (i) a free end, (ii) a fixed end, (iii) a loaded end? A uniform vertical rod of length λ has the upper end fixed. The lower end carries a wheel of inertia I. If the wheel receives an impulsive couple, discuss the resulting oscillations.

16. A uniform straight rod with one end clamped and the other end free vibrates transversely. With the notation of 7, 8, prove that the period is given by $\cos a\lambda \cosh a\lambda + 1 = 0$, and deduce that the first root is $a\lambda = 1{\cdot}87$ approximately.

17. A uniform bar vibrating transversely is subjected to a pull P. Prove that the equation of motion is

$$\frac{\partial^4 y}{\partial x^4} - \frac{P}{EI}\frac{\partial^2 y}{\partial x^2} + \frac{m}{EI}\frac{\partial^2 y}{\partial t^2} = 0.$$

Assuming that y has a factor $\cos \omega t$, prove by considering some particular mode of fixing that the effect of P is to speed-up the rate of vibration.

18. Discuss the forced vibrations of a stretched string when the mid-point has $y = \beta \cos(\omega t + \varphi)$. Take the origin at the undisplaced middle, so that the end conditions are $y = 0$, $x = \pm\frac{1}{2}\lambda$.

19. The motion of a vibrating membrane is given by the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = a\frac{\partial^2 w}{\partial t^2}.$$

Assuming w to be periodic in t, find solutions that vanish on the periphery of a square.

20. Obtain solutions of the equation

$$\frac{\partial^2 \xi}{\partial t^2} = A\frac{\partial^2 \xi}{\partial u^2} + B\frac{\partial^3 \xi}{\partial u^2\,\partial t},$$

which occurs in the theory of the vibrating piezo-electric rod.

21. The face of a semi-infinite solid is subjected to a temperature θ which varies with the time t in accordance with $\theta = A \sin \omega t$. Assuming that the amplitude of temperature variation diminishes with the depth of penetration, prove that in the notation of 7, 7,

$$\theta = Ae^{-\beta z}\sin(\omega t - \beta z), \;\; \beta^2 = \omega\rho\, k/2s.$$

[The phrase " a semi-infinite solid " is usually taken to mean a solid bounded by one plane face which is unlimited in all directions, the solid existing on one side only of the plane and to an unlimited depth, e.g. the surface of the ocean.]

22. Prove that the functions (i) $V = r^{-1}$; (ii) $V = \log(r + z)$ are solutions of Laplace's equation in three dimensions.

23. Verify that the equation $\dfrac{\partial V}{\partial t} = k\dfrac{\partial^2 V}{\partial x^2}$ has the solution

$$V = Bt^{-\frac{1}{2}}\exp\left\{-\frac{(x-a)^2}{4kt}\right\}.$$

24. Waves are propagated vertically through the atmosphere in accordance with the equation

$$c^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2} + a^2 c^2 w.$$

Find solutions of this equation.

25. Taking the equation of the surface of revolution in 7, 1 in the form $xq = yp$, prove by differentiation that the surface is not developable unless

$$s(x^2 + y^2) + py = 0.$$

26. Prove that the equation 7, 7 (ii) can be transformed to the type used in 7, 6, Ex. 2, by a substitution of the form

$$\theta = Ve^{-\nu t}.$$

27. A uniform bar of length λ is heat-insulated throughout its length and also at one end. The other end radiates into a medium at temperature zero. Prove that, in the notation of 7, 11, Ex. 4, the temperature distribution can be expressed in a cosine series dependent on the equation $c \tan c\lambda = h$.

CHAPTER VIII

The Method of Isoclinals

8. 1. It has already been noted that the vast majority of equations are insoluble and practically no progress in analytical methods of solution has been made for a century or more. The discussion has gone into other fields involving the complex variable and the theory of functions. Soluble equations find their way into textbooks for students to practise on, and an indiscrete silence is usually maintained about the vast insoluble remainder. An unfortunate result is the dissemination of the idea that an equation is necessarily soluble if only one could hit on the requisite trick. The scientific investigator is soon disilluded about this, and gets accustomed to meeting equations whose analytical solution defeats the best of mathematicians. But the matter cannot be left there. The scientist, faced with the necessity of getting some sort of solution somehow, is compelled to adopt other tactics for devising a non-analytical solution that suffices for the purpose in hand. It is in this field that most progress has latterly been made, and the various methods adopted can be classified as graphical and numerical.

It should be evident that if a relation holds between symbols representing anything but pure numbers, the various terms must be dimensionally similar. A temperature cannot be added to a velocity, and the cost of fuel cannot be squared. If we write $p + a/v^2$ where p is a pressure, a force per unit area, then p has the dimensions $ML^{-1}T^{-2}$. If v is a volume it has the dimensions L^3, and hence the constant a, whatever its numerical value, must have the dimensions ML^5T^{-2} if the plus sign is to have any significance.

A differential equation, like any other equation, must be dimensionally uniform if it is to make sense, and this may involve the presence of certain dimensional constants. And if similar quantities are expressed in the same unit, then numerical coefficients may appear; energy in the form of calories may be added to ergs of work if the mechanical equivalent of heat is present. A change in the units may apparently obliterate a constant by reducing it to unity, in which form it would not be printed; but if the idea of dimensions is retained, the constant is tacitly present. Thus in $x + x^2$ the symbols either represent pure numbers or else the expression is equivalent to $ax + x^2$ where a has unit value and the same dimensions as x, whatever they happen to be.

Once the equation is properly balanced in this sense, the idea of dimensions can be dropped and the symbols treated as pure numbers. We shall accordingly treat a differential equation as a relation between numbers in the present chapter. We at once find that a good deal of information can be obtained without solution.

Consider the equation

$$\frac{d^2y}{dx^2} + (1 + x - y^2)\frac{dy}{dx} = y + 2.$$

It is desired to know the curvature at the origin of any curve which satisfies this equation and passes through the origin at 60° to OX.

The uninspired procedure is to attempt to solve the equation, hoping to get a solution with two arbitrary constants since it is of the second order. One could expect to determine these constants from the two given conditions that when $x = 0$, $y = 0$ and $y' = \sqrt{3}$. Having now got the equation to the curve, we can find the first and second differential coefficients of y. Their values can be calculated at the origin and then substituted in the formula for the curvature.

All this is a waste of time, apart from the insuperable difficulty of solving the equation. Direct substitution gives $y'' = 2 - \sqrt{3} = 0{\cdot}268$ under the conditions $x = 0 = y$, $y' = \sqrt{3} = 1{\cdot}732$. Hence the curvature given by $\dfrac{1}{\rho} = \dfrac{y''}{(1 + y'^2)^{3/2}} = \dfrac{0{\cdot}268}{8} = 0{\cdot}0335$ approximately. The point to note is that if numerical values are given to some of the unknowns x, y, y', &c., these can be directly substituted in the differential equation, thereby providing a little further information. In practice this is frequently the only course open to us.

8, 2. Consider now the first-order, first-degree equation, which can be written $p = f(x, y)$. It represents a one-parameter family of curves. One member of the family in general passes through each point of the plane and its slope at that point is given by substituting the coordinates in $p = f(x, y)$. We assume that $f(x, y)$ has a unique value when x, y are known. No two members of the family can cross, since at any point the slope p is then uniquely determined.

These statements call for modification in particular instances, and two cases call for notice. Consider firstly the equation

$$\frac{dy}{dx} = (x^2 - y^2)^{\frac{1}{2}}.$$

The two straight lines $y = x$ and $y = -x$ divide the x, y plane into four regions. In the upper and lower of these regions we have y greater than x in absolute value, so that $(x^2 - y^2)^{\frac{1}{2}}$ is imaginary. This is an exception to the rule that one member of the family passes through each point. No member of the family intrudes into the upper and lower regions; the family is confined to the regions left and right.

In the second case, consider the equation

$$x^2\left(\frac{dy}{dx} - 1\right) + 2y^2 = 0.$$

When we attempt to find the slope of the particular member at the origin, we are faced with

$$\frac{dy}{dx} = 1 - \frac{2y^2}{x^2},$$

and the right side is indeterminate for the simultaneous values $x = 0 = y$. We can examine the origin more closely by drawing the line $y = \kappa x$. Let this swing round by varying κ till x and y become infinitesimal, then $\kappa = p$. This gives $\kappa = 1 - 2\kappa^2$, which leads to the two values $\kappa = -1$, $\frac{1}{2}$, in opposition to our assumption that p is everywhere uniquely determinate.

The equation is soluble as a homogeneous form and leads to the one-parameter family of quartics $Ax^3(x - 2y) = (x + y)$, as can be easily verified. These all pass through the origin, thus providing another exception to the rule that one single member of the family passes through each point. The renewed attempt to find the slope at the origin by putting $y = \kappa x$ in the solution and letting x become infinitesimal is now rewarded with the unique result $\kappa = -1$. But the knot would be more difficult to untie had the equation the slightly modified and less tractable form.

$$x^2\left(\frac{dy}{dx} - y - 1\right) + 2y^2 = 0.$$

In spite of these and similar exceptions, the rule holds in general that $p = f(x, y)$ represents a one-parameter family of curves of which one only passes through each point of the x, y plane and the slope at any point is uniquely determinate.

8, 3. *Isoclinals.*

We can introduce a slight modification by giving p a specific numerical value c, thus leading to a curve, not in the family, whose equation is $f(x, y) = c$. Choose any point A on this curve; then a member of the family passes through A and that member's slope there is c. This means that the curve is the locus of all points where the corresponding member of the family has the slope c. Such a curve is accordingly named an isoclinal, and if we cross it at numerous points by short parallel lines of slope c we get a first glimpse of the general trend of the family. The view can be much improved if we take several isoclinals at short intervals and put in the corresponding short lines. This creates what is known as a directional field, analogous to a field of force.

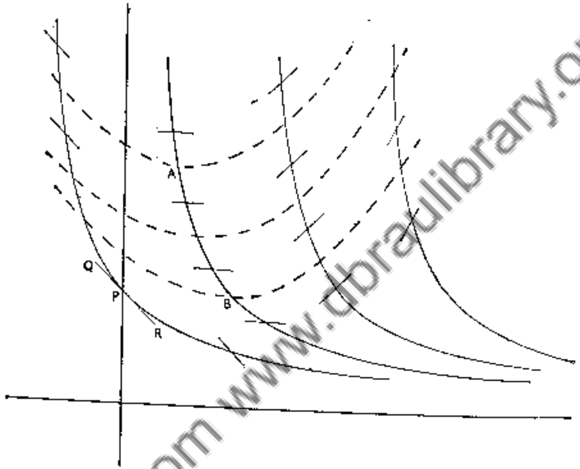

Fig. 8

Example.—The family $\frac{dy}{dx} = x - \frac{1}{y}$ has the isoclinals $y(x - c) = 1$. Putting firstly $c = 0$, we draw the rectangular hyperbola $xy = 1$, confining ourselves for convenience to the first quadrant. This isoclinal is critical in the sense that it is the locus of maxima and minima. We cross this isoclinal with a number of short lines. These will be horizontal since the assumed slope is zero. The other isoclinals are this same hyperbola displaced a distance c right or left according as c is positive or negative. We then sketch the isoclinal for some other value of c, say $c = 0{\cdot}5$, giving $y(x - 0{\cdot}5) = 1$, and cross it with short lines of slope $0{\cdot}5$. Other isoclinals can be similarly treated for $c = 1{\cdot}0, 1{\cdot}5, \ldots$ and the corresponding negative values. The members of the family that pass through A, B, &c., in fig. 8 begin to take shape and can be roughly sketched in. Each member is known as an integral curve, and in the present case it somewhat resembles a parabola with vertical axis and concavity upwards.

This crude technique can evidently be much improved by using a drawing-board. We draw a horizontal unit base from a pole O at the side. We mark positive and negative lengths 0·5, 1·0, &c., on the perpendicular at its end. The join of one of these to O gives the slope which can be transferred across the paper by set-square or parallel rulers. Having drawn to scale the requisite number of isoclinals reasonably close together, the "shorts" should be made to meet about midway between consecutive isoclinals, thus producing a sort of link polygon which can be smoothed off with a French curve.

The values $c = 0{\cdot}5$, $1{\cdot}0$, &c., may prove in some cases to be too disperse and have to be replaced by $c = 0{\cdot}1$, $0{\cdot}2$, &c. The suitable values will become apparent after a couple of isoclinals are drawn. When a specified integral curve is required, only short portions of the isoclinals need be drawn; nor need these be necessarily taken at equidistant values of c. In addition, it is frequently useful to note how the integral curves cross the co-ordinate axes by equating x or y to zero.

EXERCISES

1. Using only positive values of c, sketch the isoclinals and integral curves for the equation

$$\frac{dy}{dx} = +\sqrt{(x + y)}.$$

Note that all the family lies above the locus of minima; also a new origin on this locus leaves the equation unchanged, hence the integral curves are identical in shape and orientation. Examine how they cross the co-ordinate axes. This equation can be solved by separation of the variables (see 2, 5, No. 26), but the analytical result is not very informative for shape.

2. Sketch the isoclinals above the x axis and short portions of the integral curves for

$$2\frac{dy}{dx} = y(y + 2x).$$

The isoclinals are hyperbolas with asymptotes in common. The x axis is one of the integral curves.

3. Indicate the form of the integral curves in the first quadrant for

$$\frac{dy}{dx} = +\sqrt{(x^2 + 2y^2)}.$$

The isoclinals are similar and similarly situated ellipses.

4. Prove that all the family $\frac{dy}{dx} = +\sqrt{(x - y^2)}$ lie within a parabola. Sketch portions of a few of the integral curves starting in the fourth quadrant and ending on the x axis.

5. Using a ten centimetre unit, determine graphically the solution of $\frac{dy}{dx} = x - \frac{1}{y}$, which starts from $x = 0$, $y = 2{\cdot}4$, and terminates where $x = 0{\cdot}7$. If there is a minimum in this range, find its value and position. [$x = 0{\cdot}43$, $y = 2{\cdot}31$.]

8, 4. *Locus of Inflections.*

The shorts usually cross the isoclinals, carrying the integral curve with them; but there is the possibility that at certain places a short may be tangent to its isoclinal. We have to examine and interpret this. Consider one of the previous isoclinals, $y(x + 1) = 1$, where each short has the slope -1. The slope of the isoclinal itself varies continuously from zero at the extreme right to infinity negative at the top, so that at some point it must have the slope -1 where its short will be tangent (see fig. 8). Let P be the point of contact and let Q, R be points on the short, above and below P respectively. The values of c at Q and R are both negative and greater than unity in absolute value since the points are left of the isoclinal $c = -1$. Hence the integral curve is steeper at Q, R than at P. This means that an integral curve inflects on contact with an isoclinal.

As we move along any curve and approach an inflection, the centre of curvature recedes to infinity. It returns from infinity on the other side of the curve after we have passed the inflection, so that at the inflection itself ρ is infinite and hence $y'' = 0$. This is the usual condition for an inflection. The same thing can be proved analytically from our conception of an inflection as the contact of an isoclinal with an integral curve. Let our differential equation be $\phi(x, y, p) = 0$, so that an isoclinal is $\phi(x, y, c) = 0$, and its slope at any point is given by

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0.$$

This slope is literally the slope of the isoclinal itself and not to be confused with the slope c which characterizes the isoclinal.

If y (and hence p) was known as a function of x, substitution in the equation would produce an identity. Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} + \frac{\partial\phi}{\partial p}\frac{dp}{dx} = 0$$

for the integral curve. If dy/dx is the same in both equations, subtraction leads to

$$\frac{dp}{dx} = 0, \text{ or } \frac{d^2y}{dx^2} = 0$$

as before. The alternative possibility $\partial\phi/\partial p = 0$ need not detain us now; but there is the further condition $y''' \neq 0$.

The curve defined by $y'' = 0$ is a useful datum when expressed explicitly in terms of x, y, for it gives the locus of all points where the integral curves are inflected and touch the isoclinals. Reverting to our previous example, we have

$$\frac{dy}{dx} = x - \frac{1}{y},$$

whence
$$\frac{d^2y}{dx^2} = 1 + \frac{1}{y^2}\frac{dy}{dx} = 1 + \frac{xy - 1}{y^3},$$

so that $y'' = 0$ leads to $y^3 + xy = 1$ as the locus.

The shape is not obvious, but we can arrive at it from the following considerations:

(i) The y axis is crossed once only, at $x = 0$, $y = 1$.

(ii) For any positive x the ordinate is positive, so that the fourth quadrant is blank.

(iii) There is no value of x corresponding to $y = 0$; the x axis is never crossed.

(iv) For very small values of y we can neglect y^3. This leaves xy nearly unity, so that x is large and its sign is the sign of y. The locus is therefore doubly asymptotic to the x axis, in the first and third quadrants.

(v) Writing the locus in the form $x + y^2 = y^{-1}$ shows that for large values of y it has approximately the parabolic form $x + y^2 = 0$. The sign of $(x + y^2)$ at any point of the plane is positive or negative according as the point is outside or inside the parabola. The above form shows that y^{-1} (and therefore y) is correspondingly positive or negative. The locus therefore lies above the parabola in the second quadrant and inside the parabola in the third.

(vi) A cubic equation has either one or three real roots. Hence any ordinate meets the locus once or thrice, depending on the value of x. Regarding the locus as a cubic in y, the condition that it has two equal roots leads to $x = -3/4^{\frac{1}{3}}$ with the corresponding $y = -1/2^{\frac{1}{3}}$. The number of real values of y is then 3 or 1 according as the ordinate is left or right of this crucial ordinate.

(vii) The locus never meets the critical isoclinal $xy = 1$, and it inflects where it crosses the y axis.

These elementary considerations suffice to give a reasonably good picture of the locus; it appears dot-dash in fig. 9, where the crucial ordinate and the asymptotic parabola are also shown.

8, 5. We can now consider the differential equation more fully, and we begin by inserting the lower branches of the hyperbolic isoclinals (actually only one is inserted, to avoid encumbering the figure). Note that if $y = 0$, y' is infinite, so that the x axis is crossed vertically if at all. Also $x = 0$ gives $y' = -y^{-1}$, so that the slope is downwards or upwards according as we are above or below the origin, and it diminishes in absolute value as we recede.

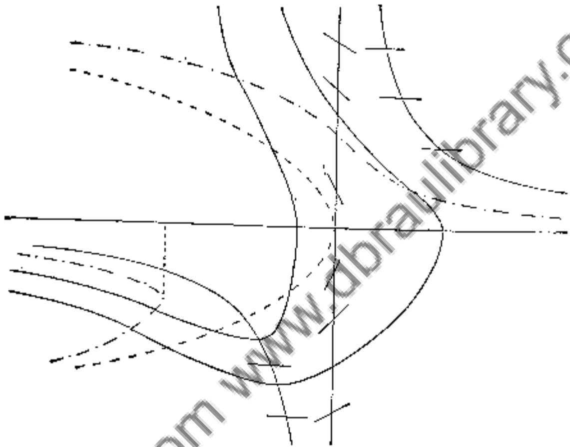

Fig. 9

We can trace another integral curve by starting at some point well down on the y axis and moving left. The curve begins to descend with gradually lessening steepness till it crosses the critical isoclinal $xy = 1$ horizontally, after which it begins to rise with increasing steepness till it meets the locus. There it inflects and turns away asymptotically to the x axis.

Returning to our starting point on the y axis, we now move to the right. The steepness of the curve continuously increases, and there being no inflections in the fourth quadrant we ultimately cross the x axis vertically. This brings us among large negative values of c, and the curve turns away to the left, inflects across the locus and proceeds upwards with increasing steepness which never becomes vertical till the curve is at infinity.

The insertion of a few more such integral curves leads to the conclusion that the family divides into two sections according as they have a minimum on the upper or lower branch of the critical isoclinal.

It is interesting to note that the apparent simplicity of the differential equation is very little augury of a simple solution.

In the foregoing analysis we have tacitly dropped the assumption that the equation is of the first degree. This calls for no special comment: $\phi(x, y, p) = 0$, being of the first order, is still a one-parameter family. If its degree in p is above the first, it simply means that more than one member of the family passes through each point, the actual number being in fact the degree in p.

EXERCISES

1. Sketch the isoclinals for the system defined by $\dfrac{dy}{dx} = x - y^2$. They are a repetition of the cricical isoclinal displaced left or right. Prove that the locus of inflections is $2y(x - y^2) = 1$, whose shape is deducible from the example worked above. Note that the slope is positive or negative according as we are inside or outside the critical isoclinal. Sketch first the integral curve through the origin, and note that (i) integral curves that cross the positive part of the y axis are asymptotic to this curve on the right; (ii) integral curves that cross the positive part of the x axis have a minimum; (iii) part of the family passes from infinity to infinity with a single inflection.

2. Rationalizing a previous exercise, sketch the isoclinals for the system

$$\left(\frac{dy}{dx}\right)^2 = x^2 + 2y^2.$$

Two integral curves now pass through each point with equal but opposite slopes. The locus of inflections can be written $8y^4 = x^2(1 - 4y^2)$, which proves that the ordinate is less than $\frac{1}{2}$ in absolute value, and approaches this value as x tends to infinity. The curve passes through the origin, and there is symmetry about both axes. Sketch the integral curves.

3. Consider the family defined by

$$\left(\frac{dy}{dx}\right)^2 = x^2 - y.$$

Note that the interior of $y = x^2$ is blank; elsewhere there are two curves through each point. There is symmetry about the y axis, and the locus of inflections is $y = -3x^2$. The integral curves which start on $y = x^2$ have a cusp there and never cross the y axis. The rest of the family cross the negative part of the y axis and pass from infinity to infinity with a single inflection.

8, 6. *Asymptotes.*

The next datum to be sought is the rectilineal asymptote. Where such exists, the slope dy/dx and the ratio y/x tend to equality as the point moves off to infinity. We therefore substitute $y = px$ in the equation and let x become indefinitely large; more correctly, we let $1/x$ become indefinitely small. Certain of the terms then become negligible in comparison with others, and their neglect may lead to an equation for p. If one such value is p_1, the presumption is that x and y jointly become infinite in the direction and ratio defined by $y = p_1 x$, $y' = p_1$.

We next try to place the asymptote more definitely by substituting $y = p_1 x + a$, $y' = p_1$. Several things may happen on letting $1/x$ become indefinitely small:

(i) The ensuing value of a may prove impossible, and we conclude that no such asymptote exists.

(ii) We may get no definite information about a. The presumption then is that any value of a will suffice and each integral curve has its own individual asymptote.

(iii) A definite value of a may be forthcoming, and we conclude that $y = p_1 x + a$ is an asymptote for the family. There is here the further possibility that the consistent relations $y = p_1 x + a$, $y' = p_1$ may satisfy the equation without x necessarily becoming infinite. In this case the line is a solution and part of the family.

The special case of horizontal asymptotes is best dealt with directly by substituting $y = a$, $y' = 0$, and seeing in what circumstances, if any, a definite value of a is forthcoming. The case of vertical asymptotes can be similarly treated by putting $x = a$, $dx/dy = 0$. The foregoing will be illustrated in the worked examples which follow. For the moment we turn to a little elementary algebra.

8, 7. *The Discriminant.*

If a polynomial $F(x)$ with literal coefficients has a repeated linear factor, the equation $F(x) = 0$ has a repeated root, and we can write

$$F(x) = (x - \alpha)^n f(x) = 0.$$

If we equate the derivative of this to zero, we get

$$F'(x) = (x - \alpha)^n f'(x) + n(x - \alpha)^{n-1} f(x) = 0.$$

These two equations are simultaneously true for the common value $x = \alpha$. The elimination of x would give a relation between the literal coefficients which is known as the discriminant, and the vanishing of this discriminant is the condition that the equation $F(x) = 0$ should have repeated roots. The simplest example is the quadratic $ax^2 + bx + c = 0$, whose discriminant is $b^2 - 4ac$.

We apply these ideas to the one-parameter family $\phi(x, y, p) = 0$, and their isoclinals $\phi(x, y, c) = 0$. Regarding this latter as an equation in c, the condition for equal roots would be found by eliminating c between

$$\phi(x, y, c) = 0 = \frac{\partial \phi}{\partial c}.$$

But it is demonstrated in texts on the differential calculus that this is the method of finding the envelope of the family $\phi(x, y, c) = 0$. We conclude that the c-discriminant includes (possibly amongst other things) the envelope of the isoclinals.

The same discriminant would be obtained by eliminating p between

$$\phi(x, y, p) = 0 = \frac{\partial \phi}{\partial p},$$

and we conclude that it may contain the envelope of the integral family. Furthermore, a cusp is a place where two values of p become equal, so we may expect the discriminant to give the cusp-locus also. Even this does not exhaust its possibilities, for it may contain the tac-locus and the nodal-locus; but for information on these matters we refer the reader to the books mentioned later.

8, 8. *Worked Examples.*

We append a number of worked examples illustrating the theory developed above.

Example 1.—Consider in greater detail a previous exercise (2, 5, No. 26) in the rationalized form

$$\left(\frac{dy}{dx}\right)^2 = x + y.$$

There are two curves with equal but opposite slopes through each point except in the half-plane below $x + y = 0$, which is blank. This line is the locus of maxima and minima; it is also the p-discriminant, giving the equal values $p = \pm 0$, and a sketch shows it is a cusp-locus. The isoclinals are the parallel lines $x + y = c^2$.

Differentiation gives $$2y'y'' = 1 + y',$$

and again $$2y'y''' + 2(y'')^2 = y''.$$

The test for inflections $y'' = 0$ involves $y' = -1$, and hence $y''' = 0$, so there are no inflections.

The search for asymptotes gives $p^2 = x + px$, or $p^2/x = 1 + p$, whence $p_1 \to -1$, as $x \to \infty$. We now put $y = -x + a$, $y' = -1$ in the equation, whence $1 = x - x + a$ and $a = 1$. We conclude that $x + y = 1$ is an asymptote; but note that it satisfies the equation identically and is therefore part of the solution.

The change of origin to any point on $x + y = 0$ leaves the equation unchanged; we conclude that the family is a repetition of a single member displaced in the north-west or south-east direction. The form of the family is shown in fig. 10.

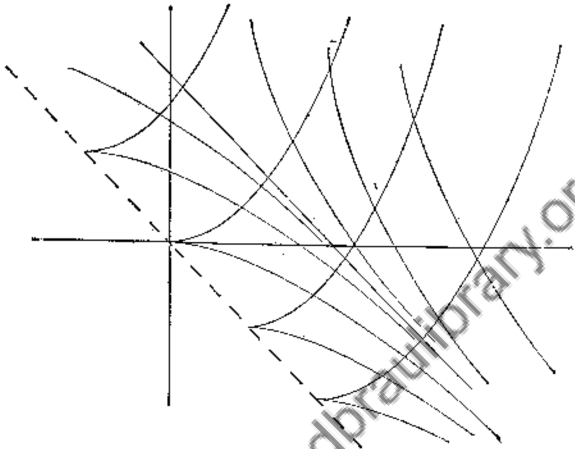

Fig. 10

Example 2.—Consider the equation

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} = 4(y - x).$$

It can be written $$(p - 1)^2 = 4(y - x) + 1,$$

which proves that the half-plane is blank below the line $y = x - \frac{1}{4}$. Elsewhere there are two members through each point. The isoclinals are the parallel lines

$$y = x + \tfrac{1}{4}c(c - 2).$$

The locus of maxima and minima is $y = x$. The search for inflexions gives

$$2pp' - 2p' = 4(p - 1),$$

or

$$2(p - 1)(p' - 2) = 0.$$

The two alternatives $p = 1$, $p' = 2$, both lead to $p'' \equiv 0$, and there are no inflexions. The search for asymptotes leads to

$$p^2 - 2p = 4(px - x),$$

or

$$p(p - 2)/x = 4(p - 1),$$

and $p_1 = 1$. The substitution $y = x + a$, $y' = 1$ leads to $0 = 4a + 1$ and $a = -\frac{1}{4}$. It looks as though $y = x - \frac{1}{4}$ is an asymptote; but the discriminant gives $p = 1$ and 1 on the isoclinal $y = x - \frac{1}{4}$, which is therefore touched by the integral curves and is their envelope. It is part of the solution. The form of the family is shown in fig. 11.

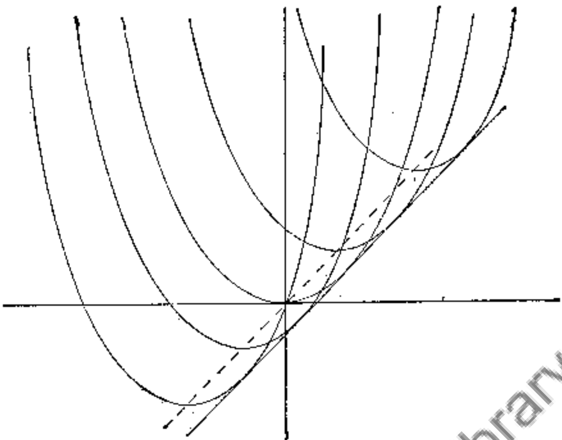

Fig. 11

Example 3.—The family is defined by

$$x^2 + 4y\frac{dy}{dx} = 4\left(\frac{dy}{dx}\right)^3.$$

There is symmetry about OX but not about OY. The isoclinals are

$$x^2 = 4c(c - y),$$

a set of vertical parabolas with a common focus at the origin; the whole of the plane is covered, with two curves at each point.

The locus of maxima-minima, $p = 0$, gives OY. The p-discriminant, condition for equal roots, gives $x^2 + y^2 = 0$, which is simply the origin, where there is a double cusp.

The search for asymptotes gives

$$x^2 + 4p^2x = 4p^3,$$

or

$$1 + \frac{4p^2}{x} = \frac{4p^3}{x^2},$$

and leads nowhere; there are no asymptotes.

Seeking for inflexions by differentiation, we have

$$2x + 4p^2 + 4p'y = 8pp'.$$

whence $p' = 0$ gives $4p^2 = -2x$. By eliminating p, we have

$$(x^2 + 2x)^2 = -8xy^2.$$

This gives (i) $x = 0$ and OY is a locus of inflexions; (ii) $x(x + 2)^2 = -8y^2$, which shows that x is negative. This part of the locus (shown dotted in the figure) has zero ordinate at the origin and at $x = -2$. Further left it opens out indefinitely like a semi-cubical parabola; there is a maximum ordinate where

$x = -2/3$. Of the integral curves that come up from the south-west, those that pass below the origin are inflected where they cross OY; the remainder inflect where they meet the locus. In fig. 12 only the lower isoclines are shown, together with three integral curves coming up from the bottom left. For completeness, the figure should be mirrored in OX.

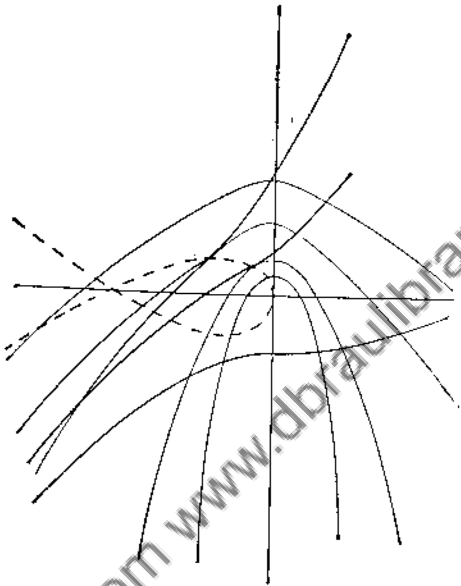

Fig. 12

EXERCISES

1. Discuss the family defined by

$$\left(\frac{dy}{dx}\right)^2 (1 - x^2)^3 = 1.$$

Prove (i) that it lies between two vertical asymptotes that are part of the solution; (ii) there are no minima-maxima, but there is a vertical locus of inflexions; (iii) the family is the repetition of a single curve displaced vertically.

2. Sketch the family defined by the equation

$$\left(\frac{dy}{dx}\right)^2 = x^2 - y.$$

Prove (i) that it is external to a certain parabola which is a locus of cusps; (ii) there is a parabolic inflexion-locus but no asymptote; (iii) there is symmetry about OY but not about OX.

3. In the family $$4\left(\frac{dy}{dx}-2\right)^2+y^2\left(\frac{dy}{dx}-1\right)=0,$$

prove that the plane is blank between two horizontal lines which are the locus of maxima-minima. These two lines are the envelope of the integral curves and part of the solution. Each member of the family has its own two asymptotes. Sketch the family.

4. Sketch the family defined by the equation

$$4\frac{dy}{dx}+(x-y)^2=4.$$

Prove that there is a maxima-minima locus but no inflexion-locus. The whole plane is covered, and the family has one asymptote in common; it is part of the solution. Each member of the family has an individual asymptote.

5. $\dfrac{dy}{dx}=y^2(2x-y).$

6. $\dfrac{dy}{dx}=x+(y^2-4)^{\frac{1}{2}}.$

7. $\dfrac{dy}{dx}=x+(4-y^2)^{\frac{1}{2}}.$

8. $(x^2-3y^2+2y)\dfrac{dy}{dx}=2x(1-y).$

9. $\dfrac{dy}{dx}+e^y=x.$

(See Miscellaneous Exercises, No. 22.)

10. $y^2\left(\dfrac{dy}{dx}\right)^2+4x\left(\dfrac{dy}{dx}\right)=4.$

11. Sketch the isoclinals and the integral curves for the equation

$$\frac{dy}{dx}+xy=1.$$

Note that the system and its inflexion-locus $(x^2-1)y=x$ are skew-symmetric. This inflexion-locus has three separate branches. One which inflects through the origin lies between $x=\pm1$. The other two are asymptotic to these lines and to the x axis in the first and third quadrant.

The equation is soluble as a linear form; but the formal solution is less revealing than the graphical work.

Two exhaustive applications of isoclinal theory are given in *Die Differentialgleichungen des Ingenieurs,* by Wilhelm Hort (Springer). The first, on water-flow, is by J. Massau, "Mémoire sur l'Intégration graphique et ses Applications". The second is by C. Cranz and R. Rothe, "Zur Lösung des Hauptproblems der äusseren Ballistik, etc."

The subject is ably treated with great detail in *Numerical Studies in Differential Equations* (Watts) by H. Levy and E. A. Baggott.

The slight but necessary knowledge of curve-tracing can be acquired from J. Edwards, *A Treatise on the Differential Calculus*; P. Frost, *Curve Tracing*; H. Hilton, *Plane Algebraic Curves*; J. L. Coolidge, *Algebraic Plane Curves*.

For a somewhat different application of semi-graphical methods, see *Journal Inst. Elec. Eng.*, LXXIX (1936), p. 362.

CHAPTER IX

Numerical Methods of Solution

9, 1. There are graphical methods for refining the solution by isoclinals; but numerical work is invariably more accurate than graphical work, nor need it take much longer, and the results are easier to check.

Numerical methods are always limited to some definite range of values and are of two types. The first type seeks to express y as a series in x, so that y can be calculated by substitution for any x in the given range. Taylor's and Picard's methods both fall in this category. The second type seeks to draw up a table of corresponding values of x and y; the methods of Runge and Euler, amongst others, fall into this class. In any case, enough initial conditions must be given to make the solution unique.

The first type is less attractive in practice than it sounds to be. It suffers from the following drawbacks: (i) the range of values for x has to be restricted to ensure the convergence of the series; (ii) the derivation of the successive terms may rapidly become complicated; (iii) the calculation of y for numerous values of x may become very tedious. All the same, this type is very useful in simple cases, and every method has its drawbacks anyway. We begin with Taylor's method as being the simplest to understand.

9, 2. *Taylor's Method.*

Taylor's theorem states that if h, k be a point on the curve $y = f(x)$, then the following expansion is valid for small values of $x - h$ under certain general conditions which usually hold:

$$y - k = (x - h)f'(h) + \frac{(x-h)^2}{2!} f''(h) + \dots .$$

This can be applied to find a curve which starts from the point h, k, and satisfies the differential equation $\phi(x, y, p) = 0$. Differentiation gives a relation involving p' which we may write $\phi_1(x, y, p, p') = 0$.

Further differentiation gives a relation involving p'' which we may write $\phi_2(x, y, p, p', p'') = 0$, and so on. The substitution $x = h$, $y = k$ in the first equation supplies p. The form ϕ_1 then supplies p', whilst ϕ_2 in turn supplies p'', and so on. As these are respectively equal to $f'(h)$, $f''(h)$ and $f'''(h)$, the coefficients in the Taylor series are now known, and the solution is relatively complete. The following is an example of the procedure:

Example.—Find the integral curve which passes through the point $x = 0$, $y = 2{\cdot}4$, and satisfies the equation

$$\frac{dy}{dx} = x - \frac{1}{y}.$$

Hence calculate its minimum ordinate and find where it is situated.

For convenience we write temporarily $2{\cdot}4\alpha = 1$. Successive differentiation then gives

$$y' = x - \frac{1}{y} \qquad = -\alpha,$$

$$y'' = 1 + \frac{y'}{y^2} \qquad = 1 - \alpha^3,$$

$$y''' = \frac{y''}{y^2} - \frac{2y'^2}{y^3} \qquad = \alpha^2 - 3\alpha^5,$$

$$y^{iv} = \frac{y'''}{y^2} - 6\frac{y'y''}{y^3} + \frac{6y'^3}{y^4} = 7\alpha^4 - 15\alpha^7,$$

and so on. The following values then ensue:

$\alpha = 0{\cdot}41667$	$\alpha^5 = 0{\cdot}01256$	$y' = -0{\cdot}4167$
$\alpha^2 = 0{\cdot}17361$	$\alpha^6 = 0{\cdot}00523$	$y'' = 0{\cdot}9277$
$\alpha^3 = 0{\cdot}07234$	$\alpha^7 = 0{\cdot}00218$	$y''' = 0{\cdot}1359$
$\alpha^4 = 0{\cdot}03014$		$y^{iv} = 0{\cdot}1783.$

The requisite series is

$$y = 2{\cdot}4 - 0{\cdot}4167x + 0{\cdot}4639x^2 + 0{\cdot}0226x^3 + 0{\cdot}0074x^4 \ldots .$$

So much for the series, for what it is worth; it remains to see what use can be made of it. The minimum value of the ordinate is obtained by equating the derivative of this to zero, i.e. from the equation

$$0 = -0{\cdot}4167 + 0{\cdot}9277x + 0{\cdot}0680x^2 + 0{\cdot}0297x^3 \ldots .$$

Limiting ourselves to these four terms, Horner's method will rapidly give $x = 0{\cdot}433$ and the substitution of this in the Taylor series gives $y = 2{\cdot}309$ to three places. This serves as a check on the working, for the reciprocal of $0{\cdot}433$ is $2{\cdot}309$, and it is known from previous work (see 8, 3, No. 5) that the minimum lies on the critical isoclinal $xy = 1$.

EXERCISES

1. Find the Taylor series for the integral curve of

$$2\frac{dy}{dx} = y(y + 2x)$$

through the point (0, 1).

[$y = 1 + 0{\cdot}5x + 0{\cdot}75x^2 + 0{\cdot}458x^3 + 0{\cdot}396x^4 \ldots$.
Note the slow convergence except for small values of x.]

2. Prove that the solution of

$$\frac{dy}{dx} + e^y = x + 1$$

through (0, 0) has the form

$$y = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30}.$$

3. The solution of

$$\left(\frac{dy}{dx}\right)^2 = x + y$$

through (1, 0) has the form

$$y = (x - 1) + 0{\cdot}5(x - 1)^2 - 0{\cdot}0833(\quad)^3 + 0{\cdot}0521(\quad)^4 \ldots .$$

Find where it meets the critical isoclinal $x + y = 0$ by putting $y = -x = t - 1$, and solving for t. This should give $t = 0{\cdot}600$ to three places, and the Taylor series then gives the calculated value $y = -0{\cdot}400$, which is correct to three places. Note the impossibility of reversing the process by starting on the line $x + y = 0$.

4. Find the solution of

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = y^2$$

as a Taylor series under the conditions $x = 0$, $y = 1$, $y' = 0{\cdot}3$.

[$y = 1 + 0{\cdot}3x + 0{\cdot}5x^2 - 0{\cdot}076x^4 \ldots$.]

5. Find the curve through $x = 0{\cdot}5$, $y = 0$ satisfying

$$\frac{dy}{dx} = x^2 + 0{\cdot}5y^2 - 0{\cdot}3.$$

6. A solution of $\frac{dy}{dx} + xy = 1$ passes through $x = 1{\cdot}5$, $y = 0$. Find the first four terms of its Taylor series.

[$y = (x - 1{\cdot}5) - 0{\cdot}75(\quad)^2 + 0{\cdot}0416(\quad)^3 + 0{\cdot}1719(\quad)^4 \ldots$.]

9, 3. *Picard's Method.*

Picard's method in its simplest form can be used to find an integral curve through h, k satisfying the equation $p = f(x, y)$. The method can be extended to equations above the first order and to simultaneous equations. The gist of the matter is iteration to give successive approximation. At the beginning of the range h to x, the ordinate y has the value k. We make the erroneous assumption that the variable y has the constant value k. This is substituted in the equation, which then takes the form $dy/dx = F(x)$, and can be integrated from h to x. The result is a new ordinate, which we may call y_1, given by an equation of the form $y_1 - k = \psi_1(x)$. This new ordinate is substituted back in the differential equation, giving it a slightly different form $dy/dx = F_1(x)$, which can again be integrated from h to x. The result is a new ordinate which we may call y_2, given by an equation of the form $y_2 - k = \psi_2(x)$. After which the process can be repeated as often as we please; it is halted when the required accuracy is attained.

The justification of the method, which will not be attempted here, involves proving the convergence of the process and demonstrating that the resulting series is a solution.

Example.—Find the solution of $\frac{dy}{dx} = x + y^2$ which passes through the origin.

As y is initially zero, we substitute this value and the equation becomes $dy/dx = x$. This leads to a new ordinate $y_1 = \frac{1}{2}x^2$, which is substituted back in the equation, whence

$$\frac{dy}{dx} = x + \frac{x^4}{4}.$$

This gives an ordinate
$$y_2 = \tfrac{1}{2}x^2 + \frac{x^5}{20},$$

which in turn leads to
$$\frac{dy}{dx} = x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400},$$

whence
$$y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400},$$

and so on. If the last term is equivalent to 10^{-5}, we have x not greater than 0·75 and the series will give five-place accuracy in the range $0 < x < 0{\cdot}75$.

When the integral curve starts from some point (h, k) other than the origin, it is better to modify the equation by transferring to a new origin at (h, k). The modified equation then runs

$$\frac{dy}{dx} = f(x + h, y + k) = f(h, k) + F(x. y),$$

where $F(x, y)$ has no independent term, so that $F(0, 0) = 0$, and $p_0 = f(h, k)$ at the new origin.

We proceed, as indicated above, by writing erroneously

$$\frac{dy}{dx} = p_0 + F(x, 0),$$

since the initial value of y is now zero. This gives a new ordinate on integration, so that $y_1 = \psi_1(x)$, and the rest follows. It is sometimes suggested that the form of the curve should be examined at the new origin as a preliminary to further work, but the practice has little to recommend it.

Example.—It is desired to find the solution of $\frac{dy}{dx} = x - y^2$ which passes through $(1, -1)$.

We begin by transferring the origin, so that

$$\frac{dy}{dx} = (x + 1) - (y - 1)^2 = x + 2y - y^2.$$

The examination of the origin is effected by assuming $y = ax^n$, where a, n are as yet unknown, and x, y are small. We immediately conclude that y^2 is negligible in comparison with y, and substitution gives

$$nax^{n-1} = x + 2ax^n.$$

Here again x^n is negligible in comparison with x^{n-1}, whatever the value of n. We are left with $nax^{n-1} = x$, so that

$$n - 1 = 1,\ n = 2,\ na = 1,\ a = \tfrac{1}{2}.$$

The approximate form of the curve at the origin is thus $y = \frac{1}{2}x^2$.

The same result is achieved more rapidly by the orthodox procedure of making the initial ordinate zero in the modified equation, which then reads $y' = x$, whence $y_1 = \frac{1}{2}x^2$ as before. On substituting this value back in the equation, we have

$$\frac{dy}{dx} = x + x^2 - \frac{x^4}{4},$$

whence

$$y_2 = \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{20},$$

which in turn is substituted back in the equation.

At this stage a glance ahead is desirable, for the equation will contain terms up to the tenth power. One step more and we shall have powers up to the twenty-third, which for ordinary purposes is too much of a good thing; in fact, many of them will be useless. We begin to impose conditions determined by the range and the limits of accuracy. Suppose we decide to stop our approximations at x^5. With this stipulation our equation reads

$$\frac{dy}{dx} = x + x^2 + \frac{2x^3}{3} - \frac{x^4}{4} - \frac{13x^5}{30},$$

whence by integration $$y_3 = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{6} - \frac{x^5}{20}.$$

The next step gives $$\frac{dy}{dx} = x + x^2 + \frac{2x^3}{3} + \frac{x^4}{12} - \frac{13x^5}{30},$$

whence $$y_4 = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{60}$$

and so on.

From a glance at the successive approximations in the present example it appears that each approximation adds to the preceding approximations one term more than is substantiated by succeeding approximations. Thus the initial $\frac{1}{2}x^2$ in y_1 is substantiated in y_2, which adds $\frac{1}{3}x^3$. This in turn is substantiated in y_3, which adds $x^4/6$, and it can be verified that y_5 will agree with y_4 to the four terms given.

What accuracy will this give, and over what range? If the last term is equivalent to 10^{-4}, the first three terms will give three-place accuracy for the range $0 < x < 0{\cdot}35$.

This would not solve the problem of finding where our integral curve meets the x axis, at $y = 1$ in the modified equation; nor would such a problem be solved in practice by taking more terms of the series. This is far too cumbersome. We should calculate the ordinate at the end of the range and then transfer the origin there. The calculation would then be repeated for a new range, and possibly again.

The success of the Picard method is obviously limited by the difficulty of performing the successive integrations. Unless these are particularly simple the method is of little use, a statement that can be substantiated by applying the method to the example which was previously solved by Taylor's method; but one should not on that account conclude that Taylor's method was necessarily superior to Picard's.

EXERCISES

1. Solve the equation $$\frac{dy}{dx} = x^2 - y^2$$

at (0, 0) and prove that three terms will give five-place accuracy for the range $0 < x < 0{\cdot}66$.

$$[y = \tfrac{1}{3}x^3 - 0{\cdot}01587x^7 + 0{\cdot}00096x^{11} \ldots .]$$

2. Examine the equation $$\frac{dy}{dx} = 3x^{-\frac{1}{2}} + y - 1$$

at (0, 1) and show that six terms will give five-place accuracy in the range $0 < x < 0{\cdot}2$.

3. Find the solution of $$\frac{dy}{dx} = x^2 + y + 3$$

through (0, 1) and prove that seven terms will give five-place accuracy over the range $0 < x < 0{\cdot}6$.

4. Verify that four terms of the solution of

$$\frac{dy}{dx} = x + y^2$$

through (0, 0) will give seven-place accuracy in the range $0 < x < 0{\cdot}5$.

5. By changing the origin, apply Picard's method to the solution of

$$\frac{dy}{dx} + xy = 1$$

through $x = 1{\cdot}5$, $y = 0$. Note how it compares with the Taylor expansion obtained in 9, 2, No. 6.

9, 4. *Euler's Method.*

The alternative to the series method which has just been discussed is a step process in which we attempt to find the ordinate-increment δy corresponding to a small abscissa-increment δx. If the initial point be (h, k), we thus achieve a new point (h_1, k_1) defined by

$$h_1 = h + \delta x, \quad k_1 = k + \delta y.$$

This point in turn is used as the starting-point for the calculation of further increments; the ultimate result is a table of corresponding values under the given initial conditions.

The difficulty in the theory of the solution is in finding a self-checking technique for the calculation of δy. In the practice of the method one meets the usual dilemma; the tedium of calculation is avoided at the cost of accuracy. It is not unknown for three expert computers to work day-in day-out for six months on end in tabulating the values for a single equation.

The simplest approach to a solution of a first-order first-degree equation $p = f(x, y)$ is due to Euler, and is based on the fundamental statement

$$p = \frac{dy}{dx} = Lt\,\frac{\delta y}{\delta x}.$$

It follows at once that δy must be approximately equal to $p\,\delta x$, the degree of accuracy depending on the smallness of δx.

Example.—For intervals of 0·1 tabulate the solution of

$$\frac{dy}{dx} = (x + y)^2$$

which starts from $(\frac{1}{2}, \frac{1}{2})$.

Initially, $$p = (x + y)^2 = (\tfrac{1}{2} + \tfrac{1}{2})^2 = 1,$$
and for an increment $\delta x = 0{\cdot}1$ we have
$$\delta y = p\,\delta x = 0{\cdot}1.$$
At the end of the increment we have
$$x = 0{\cdot}5 + 0{\cdot}1 = 0{\cdot}6 = y,$$
so that $$p = (0{\cdot}6 + 0{\cdot}6)^2 = 1{\cdot}44.$$
The next increment is $$\delta x = 0{\cdot}1, \quad \delta y = 0{\cdot}144,$$
whence $$x = 0{\cdot}6 + 0{\cdot}1 = 0{\cdot}7, \quad y = 0{\cdot}6 + 0{\cdot}144 = 0{\cdot}744.$$
For the third step $$p = (0{\cdot}7 + 0{\cdot}744)^2 = 2{\cdot}085,$$
and the increments are $$\delta x = 0{\cdot}1, \quad \delta y = 0{\cdot}2085.$$
These give $$x = 0{\cdot}8, \quad y = 0{\cdot}9525.$$
The tabulated values so far are thus:

$x = 0{\cdot}5$	$0{\cdot}6$	$0{\cdot}7$	$0{\cdot}8$
$y = 0{\cdot}5$	$0{\cdot}6$	$0{\cdot}744$	$0{\cdot}9525$.

An assessment of the accuracy can now be made; as a matter of fact, the equation is soluble by separation of the variables (2, 5, No. 25), and with the given initial conditions leads to
$$x + y = \tan\left(x + \frac{\pi}{4} - \frac{1}{2}\right).$$
The calculated value for $x = 0{\cdot}8$ is $y = 1{\cdot}0958$, so that the error is already $0{\cdot}1433$ or nearly thirteen per cent. The inaccuracy could be diminished by working with smaller increments; but the method is obviously crude and needs refinement.

9, 5. *The Euler Method, Improved.*

Consider a curve, starting from P, in which y and y' both increase with x, so that the curve rises and is concave upwards. Mark an abscissa PA and erect an ordinate AC which is cut at B below C by the tangent at P. Hitherto we have put
$$PA = \delta x, \quad \tan APB = p, \; AB = p\,\delta x,$$
and we have considered B to be a point approximately on the curve. It is apparent why the error with such a curve is cumulative.

The mean-value theorem in the differential calculus shows that at some point on the curve between P and C the tangent is parallel to PC. The point C could be located if we knew the slope of PC, which is greater than the slope at P but less than the slope at C. If we split the difference by taking the half-sum of these two latter slopes, we may expect a value which, though not necessarily correct, is better than either taken separately. The objection is, of course, that we cannot calculate the slope at C since C is not determinable; but if we make the best of a bad job by using B as equivalent to C, we can write

$$p = \tfrac{1}{2}(p_P + p_B), \quad \delta y = p\,\delta x.$$

This will give a point B_1 above B, and the process can be repeated to give another point B_2 still nearer to C. The calculation is stopped when there is no variation in the result; after which we proceed to the next increment.

Example.—Taking the same equation as before and working to four places, the slope at B, the end of the first increment, is

$$\frac{dy}{dx} = (0{\cdot}6 + 0{\cdot}6)^2 = 1{\cdot}44.$$

At the beginning the slope is $p = 1$, so we take the mean slope as

$$p = \tfrac{1}{2}(1 + 1{\cdot}44) = 1{\cdot}22.$$

For an increment $\delta x = 0{\cdot}1$, this gives

$$\delta y = p\,\delta x = 0{\cdot}122$$

and provides B_1 with $\quad x = 0{\cdot}6, \quad y = 0{\cdot}622.$

We repeat by calculating the slope at B_1 as

$$\frac{dy}{dx} = (0{\cdot}6 + 0{\cdot}622)^2 = 1{\cdot}222^2 = 1{\cdot}4933.$$

Taking the mean slope as between P and B_1, we have

$$p = \tfrac{1}{2}(1 + 1{\cdot}4933) = 1{\cdot}2467$$

and $\quad \delta y = 0{\cdot}1247, \quad x = 0{\cdot}6, \quad y = 0{\cdot}6247.$

The end-slope is now

$$(0{\cdot}6 + 0{\cdot}6247)^2 = 1{\cdot}2247^2 = 1{\cdot}4999,$$

which provides a new mean slope

$$\tfrac{1}{2}(1 + 1{\cdot}4999) = 1{\cdot}2500$$

and $\quad \delta y = 0{\cdot}1250, \quad x = 0{\cdot}6, \quad y = 0{\cdot}6250.$

Continuing, $\quad (0{\cdot}6 + 0{\cdot}6250)^2 = 1{\cdot}2250^2 = 1{\cdot}5006,$

$$p = \tfrac{1}{2}(1 + 1{\cdot}5006) = 1{\cdot}2503, \quad \delta y = 0{\cdot}1250.$$

There is no change in the increment δy between the last two calculated values; we have reached the limit of accuracy with

$$x = 0{\cdot}5, \quad 0{\cdot}6$$
$$y = 0{\cdot}5, \quad 0{\cdot}6250.$$

It can be verified from the known solution that the error here is in excess by about

2 in the third decimal place. If the solution is to be continued we commence the next increment with the initial values

$$x = 0{\cdot}6, \quad y = 0{\cdot}625, \quad p = 1{\cdot}5006.$$

The increments are then $\delta x = 0{\cdot}1, \quad \delta y = 0{\cdot}1501.$

These give

$$x = 0{\cdot}7, \quad y = 0{\cdot}7751$$

as B at the end of the new increment, so that

$$p = 1{\cdot}4751^2 = 2{\cdot}1759.$$

The first mean-slope for the new increment is

$$\tfrac{1}{2}(1{\cdot}5006 + 2{\cdot}1759) = 1{\cdot}8383.$$

The work then proceeds as before.

This modification of the Euler method is an improvement on the simpler method which precedes it; but it still suffers from the same vital defect, that it gives no indication of the order of magnitude of any error incurred. The only way to overcome this defect is to find out algebraically what the answer ought to be and then invent a technique for calculating it with errors of known magnitude.

9, 6. *Runge's Method.*

Suppose we knew that $y = F(x)$ was the solution of $p = f(x, y)$ through (h, k). We could then write

$$\frac{dy}{dx} = F'(x) = f(x, y).$$

We should also have

$$k + \delta y = F(h + \delta x) = F(h) + F'(h)\delta x + \tfrac{1}{2}F''(h)\,\delta x^2 + \ldots$$

or

$$\delta y = F'(h)\,\delta x + \tfrac{1}{2}F''(h)\,\delta x^2 + \ldots .$$

The coefficients on the right can be calculated by total differentiation with respect to x. To simplify the procedure we borrow a notation from partial differentiation by writing

$$f(h, k) = f, \quad \frac{\partial f}{\partial h} = f_1, \quad \frac{\partial f}{\partial k} = f_2, \quad \frac{\partial^2 f}{\partial h\,\partial k} = f_{12} = f_{21}, \quad \&c.$$

The total differentiation of $f(x, y)$ then gives

$$F''(x) = \frac{d}{dx}f(x, y) = \left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}\right) f(x, y),$$

or,

$$F''(h) = f_1 + f_2 f.$$

By repeating the differentiation, we have

$$F'''(h) = \left(\frac{\partial}{\partial h} + f\frac{\partial}{\partial k}\right)(f_1 + f_2 f)$$
$$= f_{11} + f_1 f_2 + 2f_{12}f + f_2^2 f + f_{22}f^2.$$

The increment δy is then given, as far as the third order terms in δx, by the relation

$$\delta y = f\,\delta x$$
$$+ \tfrac{1}{2}(f_1 + f_2 f)\delta x^2$$
$$+ \tfrac{1}{6}(f_{11} + f_1 f_2 + 2f_{12}f + f_2^2 f + f_{22}f^2)\,\delta x^3.$$

We observe that the first line here is the simple Euler method. If the above expression is used in its entirety, we know that the error begins to appear in terms of the order δx^4. What is now required is a simple technique for performing the calculation with the above expression. This technique is associated with the name of the German mathematician, C. Runge.

Notice for a start that if we calculated the slope p_1 at E, the middle of PB, we should get

$$p_1 = f(h + \tfrac{1}{2}\delta x,\ k + \tfrac{1}{2}\delta y) = f(h + \tfrac{1}{2}\delta x,\ k + \tfrac{1}{2}f\,\delta x)$$
$$= f + \tfrac{1}{2}(f_1 + f_2 f)\,\delta x + \tfrac{1}{8}(f_{11} + 2f_{12}f + f_{22}f^2)\,\delta x^2 + \ldots .$$

If we apply this slope to the range δx for calculating δy, we get a value which is correct in the δx and δx^2 terms; but it fails in the δx^3 terms. Runge remedies the deficiency by a double application of the Euler method.

He accepts the slope p at the beginning of the range and the slope p_1 at the end of the increment. He then finds a particular slope p'' at the end of the increment. By a proper choice of p'', a Simpson-rule combination $(p + 4p_1 + p'')/6$ gives the result correct as far as δx^3. The search for the appropriate p'' is conducted as follows: Taking the slope p at P, we find B from $\delta y_1 = p\,\delta x$. The slope p' at B is calculated as

$$p' = f(h + \delta x,\ k + \delta y_1)$$
$$= f(h + \delta x,\ k + p\,\delta x).$$

This slope p' is transferred to P and used to calculate a new point B' from $\delta y_2 = p'\,\delta x$. The slope p'' at B' is calculated as

$$p'' = f(h + \delta x,\ k + \delta y_2)$$
$$= f(h + \delta x,\ k + p'\delta x).$$

It can be verified that the Taylor expansions of p' and p'' are

$$p' = f + (f_1 + f_2 f)\,\delta x + \tfrac{1}{2}(f_{11} + 2f_{12}f + f_{22}f^2)\,\delta x^2 + \ldots,$$

$$p'' = f + (f_1 + f_2 p')\,\delta x + \tfrac{1}{2}(f_{11} + 2f_{12}p' + f_{22}p'^2)\,\delta x^2 + \ldots$$

$$= f + (f_1 + f_2 f)\,\delta x + \tfrac{1}{2}(f_{11} + 2f_1 f_2 + 2f_2^2 f + 2f_{12}f + f_{22}f^2)\,\delta x^2 + \ldots.$$

No use whatever is made of p' beyond its service in calculating p''. Looking at the range PA, we find we have three available slopes. They are p at the beginning, p_1 in the middle, and p'' at the end. There is no abstruse theory behind the next statement. It just happens that if we combine these in a Simpson-rule we get a slope $(p + 4p_1 + p'')/6$, which is the correct value as far as terms in δx^2. It is simple algebra to verify this. The increments are then connected by

$$\delta y = (p + 4p_1 + p'')\,\delta x/6,$$

which is correct up to δx^3. It only remains to get a simple methodical arrangement of the working.

It is found better in practice to write

$$(p + 4p_1 + p'')/6 \equiv p_1 + \tfrac{1}{3}\{\tfrac{1}{2}(p + p'') - p_1\}.$$

The steps are then:

(i) calculate p, p' and p'';
(ii) take the half-sum $\frac{1}{2}(p + p'')$;
(iii) calculate p_1 and subtract from (ii);
(iv) add a third of the difference (iii) to p_1;
(v) the increment δy is obtained by multiplying δx by (iv).

Example. Runge exemplified his method by solving

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

through (0, 1) as far as $x = 1$. For a reason which will be given later, the range was covered in three unequal increments, 0·2, 0·3 and 0·5. The equation is soluble as a homogeneous form, so the results obtained can be checked against directly calculable values. Taking as a first increment $\delta x = 0{\cdot}2$, we have

$$p = 1, \quad \delta x = 0{\cdot}2, \quad \delta y_1 = 0{\cdot}2, \quad B = 0{\cdot}2,\ 1{\cdot}2.$$

$$p' = \frac{1{\cdot}2 - 0{\cdot}2}{1{\cdot}2 + 0{\cdot}2} = 0{\cdot}7143, \quad \delta y_2 = 0{\cdot}1429, \quad B' = 0{\cdot}2,\ 1{\cdot}1429.$$

$$p'' = \frac{1{\cdot}1429 - 0{\cdot}2}{1{\cdot}1429 + 0{\cdot}2} = 0{\cdot}7021, \quad E = h + \tfrac{1}{2}\delta x, \quad k + \tfrac{1}{2}\delta y_1 = 0{\cdot}1,\ 1{\cdot}1.$$

$$p_1 = \frac{1{\cdot}1 - 0{\cdot}1}{1{\cdot}1 + 0{\cdot}1} = 0{\cdot}8333.$$

$$\begin{aligned} p &= 1 \\ p'' &= 0{\cdot}7021 \\ \text{sum} &= 1{\cdot}7021 = p + p'' \\ \tfrac{1}{2}\,\text{sum} &= 0{\cdot}8511 = \tfrac{1}{2}(p + p'') \\ p_1 &= 0{\cdot}8333 \\ \text{diff} &= 0{\cdot}0178 = \tfrac{1}{2}(p + p'') - p_1 \\ \tfrac{1}{3}\,\text{diff} &= 0{\cdot}0059 \\ \text{slope} &= 0{\cdot}8302 \end{aligned}$$

$$\delta y = 0{\cdot}8302 \times 0{\cdot}2 = 0{\cdot}1678$$

$$x = 0{\cdot}2, \quad y = 1{\cdot}1678.$$

For the next step we have

$$p = \frac{0{\cdot}3678}{1{\cdot}3678} = 0{\cdot}7076, \quad \delta x = 0{\cdot}3, \quad \delta y_1 = 0{\cdot}2123, \quad B = 0{\cdot}5,\ 1{\cdot}3801.$$

$$p' = \frac{0{\cdot}3801}{1{\cdot}8801} = 0{\cdot}4681, \quad \delta y_2 = 0{\cdot}1404, \quad B' = 0{\cdot}5,\ 1{\cdot}3082.$$

$$p'' = \frac{0{\cdot}3082}{1{\cdot}8082} = 0{\cdot}4470, \quad E = 0{\cdot}2 + \tfrac{1}{2}(0{\cdot}3),\ 1{\cdot}1678 + \tfrac{1}{2}(0{\cdot}2123)$$
$$= 0{\cdot}35,\ 1{\cdot}2740.$$

$$p_1 = \frac{0{\cdot}9240}{1{\cdot}6240} = 0{\cdot}5690.$$

$$\begin{aligned} p &= 0{\cdot}7076 \\ p'' &= 0{\cdot}4470 \\ \text{sum} &= 1{\cdot}1546 \\ \tfrac{1}{2}\,\text{sum} &= 0{\cdot}5773 \\ p_1 &= 0{\cdot}5690 \\ \text{diff} &= 0{\cdot}0083 \\ \tfrac{1}{3}\,\text{diff} &= 0{\cdot}0028 \\ \text{slope} &= 0{\cdot}5718 \end{aligned}$$

$$\delta y = 0{\cdot}5718 \times 0{\cdot}3 = 0{\cdot}1715.$$

$$x = 0{\cdot}5, \quad y = 1{\cdot}3393.$$

For the third step

$$p = \frac{0{\cdot}8393}{1{\cdot}8393} = 0{\cdot}4563, \quad \delta x = 0{\cdot}5, \quad \delta y_1 = 0{\cdot}2282, \quad B = 1{\cdot}0,\ 1{\cdot}5675.$$

$$p' = \frac{0{\cdot}5675}{2{\cdot}5675} = 0{\cdot}2210, \quad \delta y_2 = 0{\cdot}1105, \quad B' = 1{\cdot}0,\ 1{\cdot}4498.$$

$$p'' = \frac{0{\cdot}4498}{2{\cdot}4498} = 0{\cdot}1836, \quad E = 0{\cdot}75,\ 1{\cdot}4534.$$

$$p_1 = \frac{0{\cdot}7034}{2{\cdot}2034} = 0{\cdot}3192.$$

$$
\begin{aligned}
p &= 0{\cdot}4563\\
p'' &= 0{\cdot}1836\\
\text{sum} &= 0{\cdot}6399\\
\tfrac{1}{2}\,\text{sum} &= 0{\cdot}3200\\
p_1 &= 0{\cdot}3192\\
\text{diff} &= 0{\cdot}0008\\
\tfrac{1}{3}\,\text{diff} &= 0{\cdot}0003\\
\text{slope} &= 0{\cdot}3195
\end{aligned}
$$

$$\delta y = 0{\cdot}3195 \times 0{\cdot}5 = 0{\cdot}1598.$$

$$x = 1{\cdot}0, \quad y = 1{\cdot}4991.$$

It is known from the calculable value that this result is in excess by about 8 in the fourth decimal place.

The accuracy of the method depends on the negligibility of the fourth and higher powers of δx. When the curve is turning over and losing steepness, the vertical increments tend to diminish and one is justified in stepping-out horizontally; that is what happened above. Conversely, if the curve begins to rise steeply the vertical increments tend to become unreliably large unless one cuts down the horizontal steps. The above form of working is not necessarily to be adopted as standard; experience shows that individual workers modify the arrangement to their own liking. The great thing is to keep it systematized.

Several writers have suggested modifications of Runge's method. One of the neatest, due to Heun, is thus.

$$\delta y_1 = p\,\delta x \quad p' = f(h + \tfrac{1}{3}\delta x,\ k + \tfrac{1}{3}\delta y_1),$$

$$\delta y_2 = p'\delta x \quad p'' = f(h + \tfrac{2}{3}\delta x,\ k + \tfrac{2}{3}\delta y_2),$$

whence

$$\delta y = \tfrac{1}{4}(p + 3p'')\,\delta x,$$

and the error begins with dx^4.

The numerical work is inevitably lengthy if the range of values for x is at all extended. Methods of extrapolating beyond a limited range by using a table of differences have been invented to lighten the labour. None of these will be treated here; the best known is due to Adams and Bashforth. The graphical and numerical treatment of differential equations, including simultaneous and partial, now covers a wide field, and the only book specifically devoted to the matter is the work by Levy and Baggott already mentioned.

EXERCISES

1. Apply both Euler's simple method and its modified form to the equation

$$\frac{dy}{dx} = \frac{2xy}{x^2 + y^2}.$$

Assume that the solution starts from $x = 1$, $y = 2$, and extend it as far as $x = 1{\cdot}5$ in increments of 0·1. Compare your answers with the calculable result 2·4271.

2. Apply Runge's method to the above equation over the same range (i) in one step, (ii) in two steps.

3. Solve the equation $$\frac{dy}{dx} = (x + y)^3$$

by Runge's method for the initial conditions $x = 0{\cdot}5 = y$.

[$x = 0{\cdot}5$ 0·6 0·7,
$y = 0{\cdot}5$ 0·6231 0·8087,

so that the calculable error is an excess of about 3 in the fourth decimal place.]

4. It is known (see 8, 8, No. 11) that the solution of

$$\frac{dy}{dx} + xy = 1$$

through $x = 1{\cdot}5$, $y = 0$, has a maximum value. Employ Runge's method to locate it. [The following values can be obtained

$x = 1{\cdot}5$	1·7	1·9	2·1	2·3	2·5	2·7
$y = 0$	0·1712	0·2874	0·3574	0·3919	0·4011	0·3941.

The last entry shows that the maximum is passed. Incidentally, it must satisfy the condition $xy = 1$. The corresponding products for the last three entries are 0·9014, 1·0028 and 1·0421. This shows that the maximum must lie between the penultimate and antepenultimate entries. Proportional parts then give $x = 2{\cdot}490$, whence $y = 0{\cdot}4016$.]

5. It is known that the solution of

$$\frac{dy}{dx} + e^y = x$$

through 0, 0 has a minimum near $x = 0{\cdot}6$. Locate it more precisely by Runge's method.

CHAPTER X

Equations in Three Variables

10, 1. *Geometrical Preliminaries.*

Gathered together in this chapter is a number of properties of equations in three variables. The various equations are simultaneous, ordinary, or partial; but a connecting thread runs through them, and they all occur in diverse fields of applied mathematics.

Certain geometrical preliminaries are almost indispensable. Not only do they facilitate comprehension, but they indicate the nature of the solution to be expected and suggest the method of attack. They are summarily treated here, and can be amplified from texts specifically devoted to the analytical geometry of three dimensions.

The direction of a straight line in space is specified by three numbers called the direction cosines, and usually denoted by l, m, n. They are connected by the relation $l^2 + m^2 + n^2 = 1$. Any three numbers proportional to these are called direction ratios, so that the same line can have direction ratios $2 : 1 : -2$ or direction cosines $\frac{2}{3}$, $\frac{1}{3}$, $-\frac{2}{3}$. Two lines are parallel if they have the same direction cosines; the two lines specified by (l, m, n) and (l', m', n') are perpendicular if $ll' + mm' + nn' = 0$, and the same relation holds for direction ratios.

A space curve which is not plane is said to be skew. Common examples are the edge of a screw thread and the path of a smoke particle. Just as a straight line is specified by the equations of the two planes of which it is the intersection, so a skew curve may be specified as the intersection of two surfaces. These are not unique since any third surface through the common intersection would serve equally well.

If A, B be two adjacent points on a skew curve at interval ds with components dx, dy, dz, these latter are the direction ratios of the tangent at A, the actual direction cosines being dx/ds, dy/ds, dz/ds. Consequently the equation

$$X\,dx + Y\,dy + Z\,dz = 0$$

implies that the arc-element is perpendicular to the direction specified by the direction ratios $X : Y : Z$. Similarly, the equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

imply that the arc-element is parallel to this direction.

The orientation of a plane in space is specified by the direction cosines of its normal. If a surface has the equation $\phi(x, y, z) = 0$, the tangent plane at any point is perpendicular to the normal, whose direction ratios are

$$\frac{\partial \phi}{\partial x} : \frac{\partial \phi}{\partial y} : \frac{\partial \phi}{\partial z}.$$

Alternatively, if the equation of the surface is written $z = f(x, y)$, the direction ratios are

$$\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : -1.$$

These are almost invariably written more succinctly as $p : q : -1$, with the convention that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q.$$

If a straight line L meets a plane P at a point A, there are always at least two directions, starting from A and lying in P, which are perpendicular to L. If L happens to be perpendicular to P, then every such line through A is perpendicular to L. Carrying this idea over to curves and surfaces, let a curve C meet a surface S in A. Then starting from A there is always a line-element ds (in fact two) that lies on S and is perpendicular to C. If C happens to meet S normally, then every such line-element is perpendicular to C, and lies in the plane tangent to S at A.

We now consider a family of curves C such that a member of the family passes through each point of space. If an arbitrary point A_1 on S lies on one of the family which we can call C_1, we can draw ds on S perpendicular to C_1. Supposing this brings us to A_2 on what we can call C_2, we draw another ds on S perpendicular to C_2, thus reaching A_3 on C_3, and so on. In this way we can draw a curve on S which is perpendicular to every C that it meets; and as the initial A_1 was arbitrary, we can draw an infinity of such curves on S. On the other hand, not every curve, drawn at random on S, would necessarily be perpendicular to every C that it met. This would be the case only if the family C happened to be orthogonal to S. For example, every curve on a sphere is orthogonal to every radius that it meets.

We can view the matter somewhat differently by dispensing with the surface S whilst retaining the family C. Taking any point A in space, let us drop the short perpendiculars from A to every member of C in the immediate neighbourhood of A. Two alternatives then present themselves. Firstly, the plane containing all these short perpendiculars may have an envelope and so define a surface. Moreover, by varying A we get a family of such surfaces orthogonal to the family of curves C. Alternatively, the plane may have no envelope, and then no such family of surfaces exists.

The points we wish to emphasize are that, if we are given a family of space-curves, then (i) a family of orthogonal surfaces does not necessarily exist; but (ii) we can always draw on any given surface an infinity of curves which are everywhere perpendicular to the members of the family where they meet them.

10, 2. *Simultaneous Equations.*

We can now turn our attention to differential equations, and we begin with simultaneous equations, of which simple cases have been discussed in Chap. V.

Suppose we have an electrostatic field of force where X, Y, Z are the components at any point, these being functions of the co-ordinates. The simultaneous equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

state that the force acts in the direction of the line-element ds. A line for which this holds at every point is called a "line of force", and (i) are its differential equations. Similarly, if a particle of fluid in steady motion has component velocities u, v, w (these being functions of the co-ordinates), then

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

are the differential equations of the stream-lines. Equations of the same type arise if the velocities of a point moving in a plane are given by

$$\frac{dx}{dt} = f(x, y, t); \quad \frac{dy}{dt} = F(x, y, t),$$

for these can be written as

$$\frac{dx}{f} = \frac{dy}{F} = \frac{dt}{1}.$$

A line of force, or a stream-line, is usually a skew curve and accordingly is definable as the intersection of two surfaces. The problem then is to derive two cartesian equations from the differential equations (i). As pointed out previously, the pair of surfaces is not unique, so that different methods of solution may lead to seemingly disparate results.

The simplest case occurs when two of the three equations

$$\frac{dx}{X} = \frac{dy}{Y}, \quad \frac{dx}{X} = \frac{dz}{Z}, \quad \frac{dy}{Y} = \frac{dz}{Z} \quad . \quad . \quad . \quad . \text{(ii)}$$

each contain only two variables. The integrations can then be performed by the methods expounded earlier in the book. As an illustration consider the equations

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{-xy}.$$

We can write $\quad x\,dx + z\,dz = 0, \quad y\,dy + z\,dz = 0,$

of which the integrals are respectively

$$x^2 + z^2 = a, \quad y^2 + z^2 = b.$$

The curves are thus given as the intersections of two families of horizontal circular cylinders. Note that we could also write

$$x\,dx = y\,dy,$$

showing that the curves lie on the hyperbolic cylinders

$$x^2 = y^2 + c.$$

This equation is merely the difference of the previous two, and illustrates that the surfaces defining the curves are not unique.

Possibly only one of the equations (ii) contains only two variables. Let it be the first, in which case it leads to a solution $f(x, y, a) = 0$. We can now eliminate y from the second equation (or x from the third) and reach another solution $F(x, z, a, b) = 0$. The functions f and F serve to define the two families of surfaces, each depending on a single parameter.

Consider the equations

$$yz\,dx = xz\,dy = dz.$$

They can be written $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xyz}.$$

The first pair gives $y = ax$, which shows that each curve lies on a plane through the z axis. Using the first and last terms, we can eliminate y and write

$$\frac{dx}{x} = \frac{dz}{ax^2z}, \quad \text{or} \quad ax\,dx = \frac{dz}{z},$$

which leads to $$e^{ax^2} = cz^2 = e^{xy},$$

either equation serving for the second surface. It is readily verified by differentiation that these really do satisfy the original equations. Thus if we take

$$cz^2 = e^{xy}, \quad \log c + 2 \log z = xy,$$

whence $$\frac{2\,dz}{z} = x\,dy + y\,dx,$$

consistent with the original equations.

A third method of solution depends on the algebraic theorems of ratio. Given three equal fractions

$$\frac{a}{f} = \frac{b}{g} = \frac{c}{h},$$

each is equal to $$\frac{la + mb + nc}{lf + mg + nh},$$

where l, m, n are any multipliers of which one or two (but not all three) may be zero. This proposition is utilized to give a zero denominator, whence it follows that the numerator must also be zero. This supplies an integral, and the process may possibly be repeated, or a second integral may be derived by the previous method of elimination.

Consider the equations

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}.$$

Here the sum of the denominators is zero, which is equivalent to taking the multipliers l, m, n as unity. Hence

$$dx + dy + dz = 0, \quad x + y + z = a.$$

Giving l, m, n the respective values x, y, z, we again have zero as the sum of the denominators, so that

$$x\,dx + y\,dy + z\,dz = 0, \quad x^2 + y^2 + z^2 = b.$$

The curves are accordingly plane sections of spheres.

This method of multipliers is always effective when the denominators are linear, but not homogeneous. For further details as to the choice of the necessary multipliers the reader should consult Ince, *Ordinary Differential Equations*, 2·701. When the denominators are homogeneous, even though not linear, the solution can sometimes be achieved by the substitution $x = uz$, $y = vz$. Applied to the last example the working would run thus:

$$\frac{u\,dz + z\,du}{z(v-1)} = \frac{v\,dz + z\,dv}{z(1-u)} = \frac{dz}{z(u-v)}$$

$$= \frac{du}{v - 1 + uv - u^2} = \frac{dv}{1 - u - uv + v^2}.$$

The last equation contains only two variables, and is equivalent to

$$\frac{2d(u+v+1)}{(u+v+1)} = \frac{d(u^2+v^2+1)}{(u^2+v^2+1)},$$

whence $$(u+v+1)^2 = c(u^2+v^2+1).$$

In terms of the original co-ordinates this is

$$(x+y+z)^2 = c(x^2+y^2+z^2),$$

which is evidently consistent with the previous solutions.

It only remains to be added that the foregoing methods apply to equations in more than three variables; but the geometrical interpretations then break down.

EXERCISES

1. Establish that the curves defined by

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

are plane sections of the family of surfaces $xyz = c$.

2. Verify that the curves defined by

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{x+y}$$

lie on the quadrics $\quad x^2 - y^2 = a, \quad z^2 = 2(x + y + b).$

3. Prove that the equations $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$ lead to plane sections of cubic surfaces.

4. Solve the equations $$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}.$$

Give a geometrical interpretation of the result and verify that the solution includes four straight lines radiating from the origin.

5. Show that the equations $\dfrac{dx}{y-z} = \dfrac{dy}{x+y} = \dfrac{dz}{x+z}$ define plane curves and complete the solution.

6. Verify that $\varphi = x^3 + y - 3xz^2$ is a possible form of potential in that it satisfies Laplace's equation. The force being the negative potential gradient, the components are given by equations of the type

$$X = -\frac{\partial\varphi}{\partial x}.$$

Deduce that the lines of force are defined by the equations

$$\frac{dx}{3(x^2-z^2)} = \frac{dy}{1} = \frac{dz}{-6xz}$$

and lie on the surfaces $$z(3x^2 - z^2) = a.$$

7. In the case where the denominators are homogeneous (of any degree), prove that the substitution given in the text necessarily leads to an equation in two variables.

8. A point moves in a plane so that the difference of its acceleration components is constant, whilst the sum of its velocity components is proportional to the ordinate. If the point starts from rest at the origin, show that its path is defined by equations of the type

$$\frac{dx}{by+at} = \frac{dy}{by-at} = \frac{dt}{1}.$$

Calculate the ordinate at time t. Deduce that the initial motion is equally inclined to the axes in the fourth quadrant and that the ordinate is never again zero.

9. Solve the equations $\dfrac{dx}{5x+2z} = \dfrac{dy}{1} = \dfrac{dz}{4x+3z}.$

10. A curve is orthogonal to the family of surfaces $2x^2 + 3y^2 + 3z^2 = c$. If it passes through the point (1, 1, 1) prove that it is a section of the cylinder $x^3 = y^2$ by a plane through OX.

11. A set of curves is defined by

$$\frac{-dx}{xz+2y^2} = \frac{dy}{2x^2-yz} = \frac{dz}{4xy+z^2}.$$

Prove that the curves which meet the plane $z = 0$ in the circle $x^2 + y^2 = 1$ form the surface

$$x(x-z) + y(y+z) = 1.$$

10, 3. *Lagrange's Linear Equation.*

Some of the simpler cases of partial differential equations have already been treated in Chap. VII. We now propose to consider an equation that is closely allied to the ordinary simultaneous equations discussed in the last section. It is known as Lagrange's linear equation, and is traditionally written

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z),$$

or more briefly $$Pp + Qq = R. \qquad \text{(i)}$$

The allied system of simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{(ii)}$$

is known as the "subsidiary equations" of (i). We now know that their solution comprises two one-parameter families of surfaces which we can denote by

$$u(x, y, z) = a, \quad v(x, y, z) = b. \qquad \text{(iii)}$$

The intersection of two surfaces, one from each of the families (iii), gives a two-parameter curve which satisfies (ii); in connexion with (i) it is called a "characteristic". There is a characteristic through each point of space, the co-ordinates of the point serving to determine the parameters a and b in (iii).

The problem of solving equation (i) is that of finding all the values of z, as functions of x and y, which make (i) an identity. Viewed geometrically, we have to find the surfaces of which the normal at every point is perpendicular to the direction defined by $P : Q : R$ at that point. That is the geometrical interpretation of equation (i). On the other hand, the characteristic at any point has its direction given by these same ratios $P : Q : R$, and is therefore perpendicular to the aforementioned normal, so that it lies on the surface. It follows that any surface is a solution of (i) if it has characteristics lying on it, one through each point.

According to this argument the surfaces (iii) should be part of the solution since each is covered with characteristics. This is in fact the case; a proof is given further on. To get a more general solution we connect the arbitrary parameters a and b by a functional relation. If we have some definite objective in view the function has to take a specific form; otherwise we leave it arbitrary and write either

$$a = f(b), \ b = F(a), \quad \text{or} \quad \phi(a, b) = 0. \quad . \quad . \quad \text{(iv)}$$

With a specific form of function, any particular value of a (say) gives a corresponding value of b; the substitution in (iii) then gives the characteristic now corresponding to a single parameter a. By allowing a to vary, these one-parameter characteristics define a surface, a solution of (i).

The elimination of a and b gives

$$\phi(u, v) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

as the solution of (i); it is known as the "general integral". Any point satisfying (iii) must satisfy (v) in virtue of (iv).

The first of equations (iv) is equivalent to $u = f(v)$; if we give the arbitrary $f(v)$ the specific form av^0, we have $u = av^0 = a$, which substantiates what was said above about the surfaces (iii) being part of the solution.

Apart from geometrical considerations the matter can be treated analytically. Any small change of co-ordinates on $u(x, y, z) = a$ are connected by the relation

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 = u_x dx + u_y dy + u_z dz.$$

As $u = a$ is a solution of (ii), we can eliminate the line-elements and write

$$Pu_x + Qu_y + Ru_z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

Similarly

$$Pv_x + Qv_y + Rv_z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

For any surface $\phi(x, y, z) = 0$, we have

$$\phi_x + \phi_z \frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial x} = p = -\frac{\phi_x}{\phi_z}.$$

Similarly,

$$\phi_y + \phi_z \frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial y} = q = -\frac{\phi_y}{\phi_z}.$$

If ϕ is a solution of (i), the equation must be satisfied identically, and substitution gives

$$P\phi_x + Q\phi_y + R\phi_z = 0. \quad . \quad . \quad . \quad . \quad \text{(viii)}$$

The elimination of P, Q, R from (vi), (vii) and (viii) gives

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ \phi_x & \phi_y & \phi_z \end{vmatrix} = 0.$$

In works on the calculus, this determinant is called the Jacobian of u, v and ϕ; its vanishing is the condition that u, v and ϕ are functionally related. In other words, ϕ is a function of u and v, a result that accords with our geometrical treatment. Incidentally, an exactly parallel proof shows that any three solutions of (ii) must be functionally related.

The function ϕ is given a specific form when we have a definite objective. We might, for example, require a solution of (i) that passes through a certain base-curve. If this curve be defined by

$$\theta_1(x, y, z) = 0, \ \theta_2(x, y, z) = 0, \quad . \quad . \quad . \quad . \quad \text{(ix)}$$

the four equations (ix) and (iii) serve to eliminate the three co-ordinates, leaving the desired relation between a and b, for which we then substitute u and v.

Consider the problem of finding a solution of

$$(y - z)\frac{\partial z}{\partial x} + (z - x)\frac{\partial z}{\partial y} = x - y,$$

which includes the z axis. The subsidiary equations are

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}.$$

These, as we have seen previously, lead to a plane $x + y + z = a$, and a sphere $x^2 + y^2 + z^2 = b$, corresponding to $u = a$, $v = b$. The z axis is defined by $x = 0 = y$. The elimination of x and y gives $z = a$, $z^2 = b$, whence the functional relation between a and b is $b = a^2$. Substituting for a and b, we have the desired result

$$(x + y + z)^2 = x^2 + y^2 + z^2, \text{ or } xy + yz + zx = 0.$$

This is a cone with its vertex at the origin. It is easy to see that it contains the z axis; in fact, it contains all three co-ordinate axes. We can verify that the result satisfies the original equation. Differentiate with respect to x, and we have

$$y + y\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x} = 0, \ \frac{\partial z}{\partial x} = p = -\frac{y + z}{x + y}.$$

Similarly,
$$\frac{\partial z}{\partial y} = q = -\frac{z+x}{x+y}.$$

Then
$$(y-z)\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} = \frac{z^2-y^2}{x+y} + \frac{x^2-z^2}{x+y},$$
$$= x - y,$$
as it should.

As a more unusual application, consider the problem of finding surfaces orthogonal to a one-parameter family of surfaces. This is the three-dimensional analogue of orthogonal trajectories in two dimensions.

Two surfaces are orthogonal if their normals at a common point are perpendicular. If the given family is $F(x, y, z) = c$, the direction of its normal is given by the ratios

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}.$$

The direction of the normal to the surface $z = f(x, y)$ is given by $p : q : -1$. The condition of perpendicularity is

$$p\frac{\partial F}{\partial x} + q\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}.$$

The solution of this Lagrange equation gives the surfaces orthogonal to the given family.

Let the family be, for example, $x^2 + y^2 + z^2 = 2az$. The direction of the normal is given by $x : y : z - a$. The condition of perpendicularity is

$$px + qy = z - a = \frac{z^2 - x^2 - y^2}{2z}.$$

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{2z\,dz}{z^2 - x^2 - y^2} = \frac{2x\,dx}{2x^2} = \frac{2y\,dy}{2y^2} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2},$$

The first pair gives $y = mx$; the first and last give

$$x^2 + y^2 + z^2 = kx.$$

The general integral is therefore

$$x^2 + y^2 + z^2 = xf\left(\frac{y}{x}\right),$$

where the form of f is arbitrary.

EXERCISES

1. Prove that the characteristics of $px + qy = z$ are straight lines through the origin, the general integral giving all cones (not necessarily circular) with vertex at the origin.

2. If a solution of $zyp + zxq + xy = 0$ contains the straight line $x = c = y$, prove that it reduces to two planes containing OZ.

3. It was proved in 7, 1 that a surface of revolution about OZ leads to the equation $py = qx$. Reverse the process by solving this equation.

4. Consider the problem of finding the surfaces whose tangent planes all pass through a given point. Show that the problem reduces to No. 1 above.

5. The characteristics of $py - qx = a$ are helices of pitch $2\pi a$ on circular cylinders.

6. Find the surface that satisfies $yp - xq = x^2 - y^2$ and contains the line $z = 0$, $x = y$.

7. Verify that the velocity components

$$u = z(x + y), \quad v = z(x - y), \quad w = x^2 + y^2$$

are possible for an incompressible fluid in steady motion, in that they satisfy the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Deduce that the stream-lines which meet the line $x = 0$, $y = z$ lie on the surface

$$x^2 + z^2 = y(4x + y).$$

8. Prove that the surfaces orthogonal to the family $z(x^2 + y^2) = c$ are all included in

$$x^2 + y^2 - 2z^2 = f\left(\frac{y}{x}\right).$$

9. Prove that the surfaces orthogonal to the family $xz = cy$ are all included in

$$x^2 - z^2 = f(x^2 + y^2).$$

What is the form of f that leads to the particular surface $y^2 + z^2 = a$? Verify that this last surface is orthogonal to the given family.

10. In the last example worked in the text the given one-parameter family are spheres with centres on OZ and passing through the origin. Interpret the two given integrals of the subsidiary equations and satisfy yourself that they are solutions of the problem.

Any sphere through the origin, with centre on $z = 0$, is a solution of the problem. How is this covered by the general integral?

10, 4. *The Homogeneous Equation.*

Consider the problem of finding the equipotential surfaces in a field where the force at any point is perpendicular to the direction defined by $P : Q : R$, these being functions of the co-ordinates. Taking the equipotential surface as $\phi(x, y, z) = c$, the force components are

$$X = -\frac{\partial \phi}{\partial x}, \quad Y = -\frac{\partial \phi}{\partial y}, \quad Z = -\frac{\partial \phi}{\partial z}.$$

The force is accordingly perpendicular to the assigned direction if

$$P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y} + R\frac{\partial \phi}{\partial z} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

which is now the differential equation for the equipotential surface. We have already met this type of equation as (viii) in the previous section.

As written, there are actually four variables, of which ϕ is dependent on the other three. As the dependent variable is not explicitly present, and as each term contains a differential coefficient, the equation is sometimes styled "homogeneous".

There are two ways of viewing it. If $u = a$, $v = b$ are two solutions of the subsidiary equations (ii) of the last section, we know that equation (i) above holds with u and v replacing ϕ; see (vi) and (vii). The elimination of P, Q and R then leads to the vanishing Jacobian which proves that ϕ is functionally related to u and v. The solution of (i) above is accordingly $\phi = f(u, v)$, where the form of f is quite arbitrary.

Alternatively, we can view the equation three-dimensionally. Division by $\partial\phi/\partial z$ reduces it to Lagrange's form

$$Pp + Qq = R.$$

Hence the solution is $f(u, v) = 0$, where u, v are derived from the subsidiary equations and the form of f is arbitrary. Evidently the problem of solving the homogeneous equation presents no new difficulties; we append an example of its application:

Example.—Verify that the family of equipotential surfaces

$$z(x - y) = cy$$

relate to a field where the force at any point is perpendicular to the direction defined by

$$x^2 + y^2 : 2xy : z(x + y)$$

and derive the family analytically as a particular case of the general integral.

The verification is simple enough; writing the equipotentials as

$$\frac{z(x - y)}{y} = c,$$

we have the direction of the force at any point defined by

$$\frac{\partial\varphi}{\partial x} : \frac{\partial\varphi}{\partial y} : \frac{\partial\varphi}{\partial z} = \frac{z}{y} : \frac{-zx}{y^2} : \frac{x-y}{y}$$

$$= yz : -zx : y(x-y).$$

Testing whether the force is perpendicular to the assigned direction, we have

$$yz(x^2+y^2) - 2x^2yz + yz(x^2-y^2) \equiv 0.$$

For the analytical derivation we suppose that the equipotentials are $\varphi(x, y, z) = c$. The force is perpendicular to the assigned direction if

$$(x^2+y^2)\frac{\partial\varphi}{\partial x} + 2xy\frac{\partial\varphi}{\partial y} + z(x+y)\frac{\partial\varphi}{\partial z} = 0.$$

We now set up the subsidiary equations

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{z(x+y)}.$$

This leads to $\qquad \dfrac{d(x+y)}{x+y} = \dfrac{dz}{z}, \quad \dfrac{d(x^2-y^2)}{x^2-y^2} = \dfrac{dy}{y},$

and we have the integrals

$$x + y = az, \quad x^2 - y^2 = by.$$

We are at liberty to set up any relation we like between a and b; if we care to give them a constant ratio by writing $b = ac$, we have the desired result

$$z(x-y) = cy.$$

Needless to say, we can get any number of families that fulfil the conditions of the problem; we merely have to choose different relations between a and b.

EXERCISES

1. Prove that all cylinders whose generators have the direction ratios $2:3:2$ are included in $3x - 2y = \varphi(x - z)$.

In particular, the one that cuts the plane $z = 0$ in the circle $x^2 + y^2 = 4$ is

$$4(x-z)^2 + (2y-3z)^2 = 16.$$

2. A sphere has its centre at the origin. Prove that the tangent plane at any point is parallel to the direction given by

$$x(y^2-z^2) : -y(z^2+x^2) : z(x^2+y^2).$$

Establish that the surface $ax = yz$ has the same property, and find the general equation which includes all such surfaces.

10, 5. *The Total Differential Equation.*

The ordinary equation of the first order in two variables can be written $P\,dx + Q\,dy = 0$, where the coefficients are functions of the variables. Unless these coefficients are simultaneously zero the equation usually defines a unique curve through any given point; in fact, its progress can be calculated step by step. There is a technique for its integration if it is exact; in the contrary case there are integrating factors which render it exact, though they may be difficult to find.

The transition to the total equation in three variables, usually written

$$P\,dx + Q\,dy + R\,dz = 0, \qquad \text{(i)}$$

brings similarities and a marked difference. If the variables are allowed slight changes in a function $\phi(x, y, z)$, the value of the function undergoes a slight change $d\phi$ given by

$$d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz.$$

This is called a total differential. If the variables are connected by the relation $\phi(x, y, z) = c$, there is no change in the value of ϕ as we move along the surface and

$$d\phi = 0 = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz. \qquad \text{(ii)}$$

Three questions then naturally present themselves for discussion:

(i) In what circumstances can the equations (i) and (ii) be considered equivalent?

(ii) If they are equivalent, what is the technique for finding ϕ?

(iii) If they are irreconcilable, what form does the solution of (i) take?

We can approach the matter from an applied standpoint by considering a field of force. This can be specified, in any way we like, by merely postulating the force-components at each point of space. Whether such a field could be realized in practice is a different matter. We might, for example, state that the force-components are

$$P = 2z(x + y); \quad Q = 2z(x + y), \quad R = (x + y)^2.$$

The question then naturally presents itself whether there are equipotential surfaces, and if so, how to find them. The first part is readily answered; if there are level surfaces $\phi(x, y, z) = c$, the force is the negative potential gradient, and we have

$$P = -\frac{\partial\phi}{\partial x}, \quad Q = -\frac{\partial\phi}{\partial y}, \quad R = -\frac{\partial\phi}{\partial z}.$$

Consideration of the second order differential coefficients gives

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}. \quad . \quad . \quad . \quad \text{(iii)}$$

These three equations give necessary and sufficient conditions for the existence of a potential. It is readily verified, for example, that the field specified above satisfies the conditions; in fact, it derives from $z(x+y)^2 + c = 0$.

When the equations (iii) are satisfied, the equation (i) is equivalent to $d\phi = 0$, and has the solution $\phi(x, y, z) = c$; the solution is usually derivable on sight, a fact that we have not hesitated to use in the previous pages. For example,

$$y\,dx + x\,dy + z\,dz = 0$$

satisfies the tests and is plainly equivalent to

$$d(xy + \tfrac{1}{2}z^2) = 0,$$

so that the solution is $\qquad z^2 + 2xy = c.$

It may be that the three tests are satisfied but the integral still not obvious. Seeing that the equation is equivalent to $d\phi = 0$, we know that the integral is a function of the co-ordinates, and hence independent of the route followed. We accordingly make an obvious modification of the method expounded in 2, 13, and write

$$\int_a^x P(x, y, z)\,dx + \int_b^y Q(a, y, z)\,dy + \int_c^z R(a, b, z)\,dz = C,$$

where the initial point (a, b, c) is taken to suit our convenience. For example,

$$3(x^2 - z^2)\,dx + dy - 6xz\,dz = 0$$

satisfies the tests. If the integral does not appear on sight we use the integration method, taking the origin as a convenient starting-point. This gives

$$3\int_0^x (x^2 - z^2)\,dx + \int_0^y dy - 6\int_0^z 0\,dz = C,$$

whence $\qquad x^3 + y - 3xz^2 = C.$

We turn now to cases where the conditions (iii) are not satisfied. Reverting to our previously specified field of force and substituting in (i) or (ii), we have

$$2z(x+y)\,dx + 2z(x+y)\,dy + (x+y)^2\,dz = 0. \quad . \quad \text{(iv)}$$

This can be alternatively written

$$2z\,dx + 2z\,dy + (x+y)\,dz = 0, \quad . \quad . \quad . \quad . \quad \text{(v)}$$

or

$$\frac{dx}{x+y} + \frac{dy}{x+y} + \frac{dz}{2z} = 0. \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

The curious fact emerges, from a consideration of these three equivalent forms, that (iv) and (vi) satisfy (iii) whereas (v) does not. Evidently we require a more general test of integrability than (iii) provides. We achieve it by writing (i) as

$$dz = -\frac{P}{R}\,dx - \frac{Q}{R}\,dy,$$

If this is to be equivalent to $z = f(x, y)$, we have

$$dz = p\,dx + q\,dy,$$

$$-\frac{P}{R} = p = \frac{\partial z}{\partial x}, \quad -\frac{Q}{R} = q = \frac{\partial z}{\partial y}, \quad -\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{P}{R}\right) = \frac{\partial}{\partial x}\left(\frac{Q}{R}\right).$$

Remembering that P, Q and R may contain z, we have

$$\frac{1}{R}\left\{\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}\right\} - \frac{P}{R^2}\left\{\frac{\partial R}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\right\}$$

$$= \frac{1}{R}\left\{\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}\right\} - \frac{Q}{R^2}\left\{\frac{\partial R}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\right\}.$$

After performing the necessary substitutions and reductions, we reach the symmetrical result

$$P\left\{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right\} + Q\left\{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right\} + R\left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} = 0. \quad \text{(vii)}$$

This is the "condition of integrability" that decides whether a given equation is integrable or not; it evidently holds if (iii) holds, but it is more general than that. It is important to notice that if (i) be multiplied by any factor μ, a function of the variables, it would disappear in the above working, and we should be left with exactly the same test of integrability. The conclusion is that unless the test is satisfied it is useless attempting to find an integrating factor; none such exists. Those who are familiar with the language of vectors will be able to interpret the result; it states that the vector defined by the coefficients is perpendicular to its curl. The following illustrates the application:

Example.—In the steady motion of an incompressible fluid the stream-lines are given by

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}.$$

Show that the motion must be rotational.

The denominators give the ratios of the velocity components, and if the motion is irrotational, these derive from a velocity potential φ, so that

$$u = \mu z(x+y) = -\frac{\partial\varphi}{\partial x},$$

with two similar equations for v and w. Substitution in (i) gives

$$z(x+y)dx + z(x-y)dy + (x^2+y^2)dz = 0$$

with the proportionality factor μ omitted. The application of the integrability test gives

$$z(x+y)(x-3y) + z(x-y)(x-y) + (x^2+y^2)0.$$

As this is not identically zero the test fails and the equation is not integrable. No such velocity potential exists, and the motion is rotational.

10, 6. *Mayer's Method of Integration.*

We now come to the problem of integrating the total equation (i) when the condition of integrability (vii) is satisfied. Many of the simpler cases can be solved by a little rearrangement. It is readily verified, for example, that the equation

$$(x\,dy + y\,dx)(k-z) + xy\,dz = 0$$

is integrable. If we rewrite it as

$$\frac{d(xy)}{xy} + \frac{dz}{k-z} = 0,$$

it evidently has the solution

$$cxy = k - z.$$

There are three methods in common use for the less simple cases. Curiously enough, none of them attempts to find the integrating factor; it suffices to know that it exists. We propose to expound Mayer's method as being the simplest to understand; the other two methods can be found in Forsyth.

Knowing that there is a solution $\phi(x, y, z) = k$, we choose the surface that passes through a point A with co-ordinates (a, b, c). This fixes the value of k, and we have $\phi(x, y, z) = \phi(a, b, c)$. If P is the general point (x, y, z), we can calculate the ordinate at P by finding its change as we move from A to Q, where Q is on the same vertical as P and on the same level as A, so that its co-ordinates are (x, y, c). The answer will be the same whatever route we use from A to Q, so we naturally choose the simplest. This is usually the straight line $y - b = m(x - a)$, so that $dy = m\,dx$. We can now eliminate y and dy from the given equation, leaving it in the form $M\,dx + N\,dz = 0$. This will have some such solution as $f(x, z, m) = f(a, c, m)$, and by eliminating m we reach

$$f\left(x, z, \frac{y-b}{x-a}\right) = f\left(a, c, \frac{y-b}{x-a}\right)$$

as the solution.

In practice it usually suffices to take the point A as $(0, 0, c)$, so that c is the intercept on OZ. There is just a possibility that difficulty may arise through the occurrence of a zero denominator, or for some similar reason; but this can be avoided by working in a vertical plane, or by a suitable choice of A.

As an illustration of the method we take a previous equation

$$2z\,dx + 2z\,dy + (x + y)\,dz = 0.$$

The substitutions $y - b = m(x - a)$, $dy = m\,dx$

give $$2z(m + 1)\,dx + \{(m + 1)x + b - ma\}\,dz = 0.$$

It appears that $(0, 0, c)$ is not a suitable position for A as we should then lose m completely. We accordingly take $(0, 1, c)$ and reach

$$\frac{2(m + 1)\,dx}{(m + 1)x + 1} + \frac{dz}{z} = 0,$$

whence $$z\{(m + 1)x + 1\}^2 = c = z(x + y)^2.$$

The solution agrees with the result stated previously. An alternative treatment would be to take A as $(a, 0, 0)$ and work in a vertical plane by making the substitutions $y = mz$, $dy = m\,dz$. These lead to

$$2z\,dx + (3mz + x)\,dz = 0,$$

a homogeneous equation that can be solved by the usual substitution $x = uz$. Thus

$$\frac{dx}{3mz + x} = \frac{dz}{-2z} = \frac{u\,dz + z\,du}{(3m + u)z} = \frac{du}{3(m + u)},$$

whence $$(m + u)^2 z^3 = c = z(x + y)^2$$

as before.

10, 7. *The Non-integrable Case.*

It was pointed out that equation (i) is equivalent to the two simultaneous equations

$$\frac{\partial z}{\partial x} = -\frac{P}{R}, \quad \frac{\partial z}{\partial y} = -\frac{Q}{R}.$$

If P, Q, R be assigned at random, the relation $\partial p/\partial y = \partial q/\partial x$ is hardly likely to be satisfied, and then the equation cannot lead to a single solution. This would certainly be the case with such an equation as

$$x\,dy - y\,dx + dz = 0.$$

The opinion was once held that such an equation was meaningless; but equation (i) can be interpreted geometrically as meaning that the line-element ds is perpendicular to the direction defined by the ratios $P:Q:R$. A curve fulfilling these conditions can be drawn through each point on any surface we like to choose. The totality of such curves is the solution. What we cannot do is to find a family of surfaces orthogonal to the given direction; this is the meaning of non-integrability.

As an illustration we can use the above equation and find all the integral curves that line in the plane $2x - y - z = 1$.

By differentiation $\quad 2\,dx - dy - dz = 0.$

By addition $\quad (2 - y)\,dx + (x - 1)\,dy = 0.$

This solves to $\quad y - 2 = m(x - 1),$

a one-parameter family of planes. Any line where this family meets the given plane is an integral curve.

The general procedure is to eliminate one variable and its differential. The resulting equation in two variables leads to cylinders; their intersections with the given surface are the integral curves.

10, 8. *Summary.*

When faced with a total equation whose solution is not fairly evident, we begin by calculating the three brackets in (vii). If these are all zero we have an exact differential, and if the worst comes to the worst we can solve by the route method, using three integrations. When the three brackets are not all zero we apply test (vii). If this is satisfied, we know there is an integrating factor; but instead of finding it we apply Mayer's method. If test (vii) does not hold we know that there is no single solution; but solutions can be found, to lie on any given surface.

It ought to be added that though the necessity for (vii) has been proved, no attempt was made to prove its sufficiency as it was considered too difficult for the present work.

EXERCISES

1. The stream-lines for the steady motion of an incompressible fluid are given by

$$\frac{dx}{x^2 - y^2} = \frac{dy}{2yz} = \frac{-dz}{z(2x + z)}.$$

Prove that this is vortex motion.

2. The lines of force in a field are given by

$$\frac{dx}{3y} = \frac{dy}{z - 3y} = \frac{dz}{x}.$$

Prove that there are no equipotential surfaces.

3. Verify that the equation

$$(y + z)dx + (z + x)dy + (x + y)dz = 0$$

is solvable without an integrating factor, and integrate it.

4. Show that the equation

$$(y^2 + 2x + 2xz)dx + 2xydy + x^2dz = 0$$

becomes integrable by merely rearranging the terms, and integrate it.

5. The lines of force in a field are given by

$$\frac{dx}{y^2} = \frac{dy}{z} = \frac{dz}{-y}.$$

Demonstrate the existence of a potential and find the level surfaces.

6. A two-parameter family of curves is defined as the intersections of the two families of surfaces

$$xy = a, \quad xz = b.$$

Prove that they have an orthogonal family of surfaces

$$c + x^2 = y^2 + z^2.$$

(In this example and the next, take the total differentials and solve the resulting simultaneous equations for dx, &c.)

7. If the lines of force are the intersections of

$$y = ax^2, \quad y^2 = bxz,$$

show that the level surfaces are

$$x^2 + 2y^2 + 3z^2 = c.$$

8. Solve the equation $(y + z)dx + dy + dz = 0$
(i) by Mayer's method, (ii) by rearrangement.

9. Verify that the condition of integrability is satisfied by

$$y(y + z)dx + (2y + z)dy + ydz = 0$$

and solve the equation. $[e^x y(y + z) = c.]$

10. Show that the equation

$$2y\,dx + dy - dz = 0$$

is not integrable. Show that the integral curves on the surface $xy = z$ lie in vertical planes.

11. When the coefficients are homogeneous of the same degree, the work can sometimes be shortened by the substitutions

$$x = uz, \quad y = vz.$$

Apply this method to the equations

(i) $z(2x - y)dx + z(2y - x)dy = (x^2 - xy + y^2)dz.$

(ii) $y(y + z)dx + z(x + z)dy = y(x - y)dz.$

CHAPTER XI

Variable Coefficients

11, 1. *Variable Coefficients.*

The remaining type of equation that one is likely to encounter is the linear equation whose coefficients are functions of the independent variable. This opens up a prospect so extensive that no comprehensive treatment is possible in a single volume. There is no longer any technique for finding the complementary function or the particular integral. Such equations are not usually soluble in elementary terms, and in fact they generally define some transcendental function. The result is that we get whole books devoted to the ramifications of a single equation, so that there are treatises on Bessel functions, Legendre functions, Hermitian polynomials, and so on.

In spite of this impressive scene there are certain useful aspects that can be treated by elementary means. This chapter will be devoted to their consideration, for the most part limiting ourselves to the second order, which means that we consider equations of the type

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} f(x) + yg(x) = h(x), \quad . \quad . \quad . \quad . \quad \text{(i)}$$

the reduced, or auxiliary, equation being

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} f(x) + yg(x) = 0. \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

11, 2. *Reduction of the Order.*

The first proposition we shall consider is that, if a solution of the reduced equation is known, the order can be reduced by unity.

A solution of the reduced equation can sometimes be found by inspection or trial. If the equation is of the second order, the reduction is to the first order; and as the equation is linear, a solution (at least formally) can be reached by the method of 2, 17.

We suppose then that $u(x)$ is a solution of 11, 1 (ii), so that

$$\frac{d^2u}{dx^2} + \frac{du}{dx} f + ug = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The substitution $y = uv$ is equivalent to

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}, \quad \frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + u\frac{d^2v}{dx^2},$$

and if these be employed in 11, 1 (i), we have

$$(u''v + 2u'v' + uv'') + (u'v + uv')f + uvg = h.$$

This may be rearranged as

$$(u'' + u'f + ug)v + (2u' + uf)v' + uv'' = h. \quad . \quad . \quad \text{(ii)}$$

In virtue of (i) the first bracket disappears, and we are left with

$$(2u' + uf)v' + uv'' = h,$$

or
$$w' + \left(\frac{2u'}{u} + f\right)w = \frac{h}{u}, \quad v' = w. \quad . \quad . \quad . \quad \text{(iii)}$$

This last equation is linear of the first order in w, and our statement as to the lowering of the order is substantiated. Moreover, as u is presumed known, w can be found formally by the use of an integrating factor. This in turn gives v, and hence y. The determination of w from (iii) supplies one arbitrary constant; the further integration for v supplies a second, which is all that is required, and the solution may be considered complete.

We can exemplify the argument by attempting to solve the equation

$$2xy'' - xy' + y = e^x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

The reduced equation is
$$2xy'' - xy' + y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

and inspection shows it to be satisfied by $y = x$. We accordingly make the substitution $y = xv$, which involves

$$y' = xv' + v, \quad y'' = xv'' + 2v'.$$

The equation (iv) then becomes

$$2x^2v'' + x(4 - x)v' = e^x,$$

or
$$w' + \left(\frac{2}{x} - \frac{1}{2}\right)w = \frac{e^x}{2x^2}, \quad w = v'.$$

The integrating factor is

$$J = \exp\int\left(\frac{2}{x} - \frac{1}{2}\right)dx = \exp(2\log x - \tfrac{1}{2}x) = x^2e^{-\frac{1}{2}x}.$$

On multiplying both sides by this factor, we have

$$\frac{d}{dx}(wx^2e^{-\frac{1}{2}x}) = \tfrac{1}{2}e^{\frac{1}{2}x},$$

whence $$wx^2e^{-\frac{1}{2}x} = e^{\frac{1}{2}x} + a,$$

$$w = (e^x + ae^{\frac{1}{2}x})\,x^{-2} = \frac{dv}{dx}.$$

Thus $$v = \int \frac{e^x + ae^{\frac{1}{2}x}}{x^2}\,dx + b,$$

and finally, $$y = xv = x\int \frac{e^x + ae^{\frac{1}{2}x}}{x^2}\,dx + bx.$$

The integration cannot be performed in finite terms, and we have to be content with this as the solution, giving y as a transcendental function of x. On examining it we notice that the last term bx is really our original solution, found by inspection. Since a and b are both arbitrary, we can take them as zero. We conclude that

$$y = x\int x^{-2}e^x\,dx$$

is a solution, and as it contains no arbitrary constants it is a particular integral. It is left to the reader to verify that it really does satisfy the original equation.

EXERCISES

1. Solve the equation $xy'' + (1 - x)y' + y = 0$ after first verifying that $y = 1 - x$ is a solution.

2. Verify that $y = e^{-x}$ is a solution of

$$y''(1 - \tan x) + 2y' + y(1 + \tan x) = 0,$$

and hence deduce the full solution.

11, 3. *Relation between Solutions.*

It is known from the theory of our subject, and has been assumed throughout this work, that the reduced linear equation of the second order necessarily has two linearly independent solutions, which we may call y_1 and y_2. It is further known, and is proved in the appendix, that any third solution y_3 necessarily has the form $y_3 = \lambda y_1 + \mu y_2$, where λ and μ are absolute constants. This gives a certain elasticity to the choice of the two fundamental solutions; thus $ay_1 + by_2$ and $cy_1 + dy_2$ are independent so long as $ad \neq bc$, i.e. so long as the one is not a mere numerical multiple of the other. For example, in the simple oscillation equation $(D^2 + \omega^2)y = 0$ we have as solutions $\sin\omega x$, $\cos\omega x$, $a\sin(\omega x + \alpha)$, $b\cos(\omega x + \beta)$, and any two of these are independent, whilst any third is a linear combination of them.

The independence of y_1 and y_2 is not quite so absolute as one might imagine. There are, in fact, very close connexions between them. It is these connexions that we now propose to investigate. Using the form 11, 1 (ii), we have by hypothesis

$$y_1'' + y_1'f + y_1g = 0,$$
$$y_2'' + y_2'f + y_2g = 0.$$

The elimination of $g(x)$ gives

$$(y_1''y_2 - y_2''y_1) + (y_1'y_2 - y_2'y_1)f = 0.$$

It will be noticed that the former of the two brackets is the differential coefficient of the latter. Hence, if we put the latter equal to j, we can write

$$\frac{dj}{dx} + jf = 0.$$

This leads to $\quad j = c\exp\{-\int f(x)\,dx\} = y_1'y_2 - y_2'y_1, \quad . \quad . \quad . \quad$ (i)

where c is some constant.

As an illustration let us consider the reduced equation 11, 2 (v). Writing it as

$$\frac{d^2y}{dx^2} - \tfrac{1}{2}\frac{dy}{dx} + \frac{y}{2x} = 0,$$

we have $\quad f(x) = -\tfrac{1}{2}, \quad \exp\{-\int f(x)dx\} = e^{\frac{1}{2}x}.$

Hence, even if we could not solve the equation we should still be in a position to state that any two linearly independent solutions y_1 and y_2 must be connected by the relation

$$y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} = ce^{\frac{1}{2}x}.$$

The value of c will of course depend on the particular forms adopted for y_1 and y_2, in the sense that if y_1 is a solution, so is any numerical multiple of it.

The relation (i) can be thrown into a different form. If we divide by y_2^2, we have

$$\frac{y_1'y_2 - y_2'y_1}{y_2^2} = \frac{d}{dx}\left(\frac{y_1}{y_2}\right) = \frac{c\exp\{-\int f(x)dx\}}{y_2^2}.$$

Thus $$y_1 = cy_2\int\frac{\exp\{-\int f(x)dx\}}{y_2^2}\,dx, \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

and it appears that, in spite of their linear independence, the one solution is derivable from the other.

We can test our conclusion by our previous example 11, 2 (v).

Knowing that x is a solution, we can put $y_2 = x$, and by virtue of (ii) we deduce

$$y_1 = cx\int \frac{e^{kx}}{x^2}\,dx.$$

The result corresponds to the coefficient of a in the original solution.

EXERCISES

1. If P and Q are solutions of

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$$

prove that

$$PQ' - P'Q = \frac{a}{1 - x^2}.$$

2. Prove that any two linearly independent solutions of the equation

$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x + y\sin^2 x = 0$$

are connected by the relation

$$y_1y_2' - y_2y_1' = a\cos x.$$

11, 4. *The Normal Form.*

It is an advantage to have the middle term of the second order reduced equation removed for the discussion of certain points. The equation is then said to be in the normal form. Taking the reduced equation in the form 11, 1 (ii) and making the substitution $y = uv$, we have from 11, 2 (ii)

$$(u'' + u'f + ug)v + (2u' + uf)v' + uv'' = 0.$$

The middle term can be removed if u be so chosen that

$$2u' + uf = 0. \qquad \text{(i)}$$

This involves $\quad u'' = -\tfrac{1}{2}uf' - \tfrac{1}{2}u'f = -\tfrac{1}{2}u(f' - \tfrac{1}{2}f^2),$

$$u'' + u'f + ug = u(g - \tfrac{1}{2}f' - \tfrac{1}{4}f^2).$$

The equation can thus be written

$$\frac{d^2v}{dx^2} + Iv = 0, \quad I = g - \tfrac{1}{2}f' - \tfrac{1}{4}f^2, \qquad \text{(ii)}$$

a result which is independent of u. For the determination of u, we have from (i)

$$u = \exp\{-\tfrac{1}{2}\textstyle\int f(x)\,dx\}.$$

As an illustration we take our previous equation 11, 2 (v). With

$$f(x) = -\tfrac{1}{2}, \quad g(x) = \frac{1}{2x},$$

we have
$$I = \frac{1}{2x} - \frac{1}{16}.$$

The equation could accordingly be written

$$\frac{d^2v}{dx^2} + \left(\frac{1}{2x} - \frac{1}{16}\right)v = 0,$$

where
$$y = uv, \quad u = \exp\left(\frac{x}{4}\right).$$

It is as well to point out that the normal form is rarely an aid to solution; its utility lies in other directions. Incidentally, it is more general than the equation whence it derives, so that different equations may come under the same normal form. It is easily seen that this must be so; for if I is given, we can assign f arbitrarily, and so deduce g.

Taking our previous working as an illustration, put for a change

$$f = \frac{1}{x}, \quad f' = -\frac{1}{x^2}.$$

From (ii) we derive
$$g = -\frac{x^2 - 8x + 4}{16x^2},$$

and the same normal form covers the equation

$$16x^2y'' + 16xy' - (x^2 - 8x + 4)y = 0.$$

EXERCISES

1. Prove that the equation $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$

has the normal form
$$\frac{d^2v}{dx^2} = \tfrac{1}{4}v(x^2 + 6),$$

where
$$v = y \exp\left(\frac{x^2}{4}\right).$$

2. Prove that the equation of damped oscillations

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + (a^2 + b^2)x = 0$$

has the same normal form as when the oscillations are undamped.

3.
$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{1}{4x^2}\right)y = 0.$$

Here is a form of Bessel's equation. Reduce it to the normal form and draw your own conclusions from the result.

11, 5. *Zeros of the Solution.*

Our inability to solve a given equation does not wholly debar us from gleaning information about the solution. One most desirable piece of information is usually whether the solution has any zeros; in geometrical language, whether the graph crosses OX. In the event of there being any such zeros, our curiosity prompts us to seek whether they are at all numerous; and if so, whether their disposition shows any tendency to regularity. In addition, it is often useful to know approximately how the function behaves for large values of the variable. The answers to these questions are by no means always forthcoming; but there are certain propositions which are helpful and which possess an interest on their own account.

We begin with the rather remarkable proposition that no two linearly independent solutions of the same normal equation can have a common zero. We take the normal equation in the form 11, 4 (ii), and suppose that v_1 and v_2 are two linearly independent solutions, so that the one is not a mere numerical multiple of the other. The proposition is then established by showing that the contrary assumption leads to a contradiction.

We have, by hypothesis,

$$v_1'' + Iv_1 = 0 = v_2'' + Iv_2.$$

On eliminating I these give

$$v_1''v_2 - v_2''v_1 = 0 = \frac{d}{dx}(v_1'v_2 - v_2'v_1),$$

whence by integration $\qquad v_1'v_2 - v_2'v_1 = c,$

where c is some absolute constant. If we now contradict the proposition by asserting that v_1 and v_2 can be simultaneously zero, we have c of necessity zero. The last equation can then be written $v_1'/v_1 = v_2'/v_2$, which merely leads to $v_1 = av_2$; the one is a mere numerical multiple of the other and they are not independent, thus invalidating the hypothesis. As applied in practice, there are of necessity two independent solutions of the normal equation, and if the one solution passes through the origin the other one does not.

Another interesting proposition is that the zeros of the two solutions interlace; otherwise expressed, between two zeros of either solution there is a zero of the other solution.

Having already established that

$$v_1'v_2 - v_2'v_1 = c,$$

we can write $$\frac{v_1'v_2 - v_2'v_1}{v_2^2} = \frac{c}{v_2^2} = \frac{d}{dx}\left(\frac{v_1}{v_2}\right).$$

Presuming that α, β are two consecutive zeros of v_1, let us suppose that v_2 does not vanish between them. Then v_1/v_2 is a continuous function in the range $\alpha < x < \beta$, and integration gives

$$\left[\frac{v_1}{v_2}\right]_\alpha^\beta = c\int_\alpha^\beta \frac{dx}{v_2^2}.$$

The value as calculated from the left is zero, since v_1 vanishes at both end-points and v_2 vanishes at neither, by our previous proposition. But the integrand on the right does not éven change sign in the range, so that the integral cannot be zero. This contradiction forces us to the conclusion that v_1/v_2 is not a continuous function, and therefore v_2 vanishes somewhere between α and β. It is immaterial which we call v_1 and which v_2, so that the proposition is established. All of which, of course, depends on one of the solutions having two zeros. It may have only one, or even none.

As a simple illustration consider the oscillation equation $(D^2 + n^2)y = 0$. It is in the normal form, and has the two independent solutions $\sin nx$, $\cos nx$. These have no zero in common and their zeros interlace.

EXERCISES

1. Without using the normal form, prove that two linearly independent solutions of the reduced second-order equation cannot have a common zero.

2. If α, β are two consecutive zeros of v', and if I is positive throughout the interval $\alpha < x < \beta$, prove that v has a zero between α and β. Compare this with Rolle's theorem in the calculus.

11, 6. *The Comparison Theorem.*

The most fruitful method of deriving information about the location of the zeros is by making comparison with another and simpler equation in the normal form. Suppose we are concerned with an intractable equation

$$v'' + Iv = 0.$$

From a second equation $$u'' + Ju = 0$$

of our own choosing we derive

$$v''u - u''v = (J - I)uv = \frac{d}{dx}(v'u - u'v).$$

If α, β be two permissible limits of integration, we have

$$\Big[v'u - u'v\Big]_{\alpha}^{\beta} = \int_{\alpha}^{\beta} (J - I)uv\,dx.$$

If further α, β are two consecutive zeros of u, we have

$$u'(\alpha)v(\alpha) - u'(\beta)v(\beta) = \int_{\alpha}^{\beta} (J - I)uv\,dx.$$

We now propose to show that if I is greater than J throughout the range $\alpha < x < \beta$, then $v(x)$ must have a zero in this range.

Since $u(x)$ vanishes at α and β, it retains its sign between them, and there is nothing lost in taking this sign to be positive; the contrary assumption merely changes the sign throughout the argument. We accordingly have $u'(\alpha)$ positive at the left and $u'(\beta)$ negative at the right, as a rough sketch will show. As I is presumed greater than J, inspection shows that v cannot retain its sign unchanged throughout the range. For if we presume it to be positive, the integrand is always negative, whereas both terms on the left are positive; and conversely. It follows from this contradiction that in these circumstances $v(x)$ must change sign at least once in the range $\alpha < x < \beta$.

As an illustration, suppose we are faced with the equation

$$\frac{d^2v}{dx^2} + (2 - \sin x)v = 0.$$

There is no guarantee that an equation has a periodic solution merely because its coefficients are periodic, and the given equation is not particularly easy to solve. We make comparison with the equation

$$\frac{d^2u}{dx^2} + u = 0,$$

which has the solution $u = \sin(x - \alpha)$ with consecutive zeros at α, $\alpha + \pi$. As $I = 2 - \sin x$ and $J = 1$, we have $J - I = \sin x - 1$, which is certainly not positive in any part of the range $\alpha < x < (\alpha + \pi)$. As α is quite arbitrary it follows that any solution of the given equation has at least one zero in every part of OX of length π.

EXERCISES

1. Prove that any solution of Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

must have an infinity of zeros.

2. Any solution of

$$xy'' + 2y' + (1 + x)y = 0$$

has an infinity of zeros whose interval ultimately tends to π.

3. Using the notation of the text, if u and v have a common zero at $x = \alpha$ and I is greater than J, then with increasing x, v must vanish again before u.

11, 7. *Recurrence Formulæ.*

It not infrequently happens that a differential equation contains a parameter, which we may denote by n. The solution is then a function both of n and of the independent variable. It is customary in such cases to use a distinctive letter to denote the function, and the parameter is written as a suffix and called the order. For example, in Bessel's equation mentioned above, one of its solutions is denoted universally by $J_n(x)$. The independent variable is sometimes omitted when not in doubt.

It generally happens in such cases that a simple algebraic relation connects any three functions of consecutive orders. In the present instance it happens to be

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)$$

connecting the functions of orders $n - 1$, n and $n + 1$. Such a relation is known as a recurrence formula and is of great utility in tabulating the function. The recurrence formula cannot be determined until some form of the solution is known, so that it is not directly deducible from the differential equation. Conversely, the differential equation is not directly deducible from the recurrence formula; there is usually an intermediate step which is differential in form. In the present instance it is

$$J_{n-1}(x) - J_{n+1}(x) = 2 \frac{d}{dx} J_n(x).$$

EXERCISES

1. Weber's function $D_n(x)$ has the recurrence formula

$$D_{n+1} - xD_n + nD_{n-1} = 0$$

and satisfies the differential relation

$$D_n' + \tfrac{1}{2}xD_n - nD_{n-1} = 0.$$

Deduce that it is a solution of the differential equation

$$4y'' + y(4n + 2 - x^2) = 0.$$

2. Laguerre's function $L_n(x)$ satisfies the differential equation

$$xy'' + (1 - x)y' + ny = 0,$$

and the differential relation

$$L_n' - nL_{n-1}' = -nL_{n-1}.$$

Prove that its recurrence formula is

$$L_{n+1} - (2n + 1 - x)L_n + n^2L_{n-1} = 0.$$

11, 8. *Orthogonal Functions.*

The functions defined by two equations which merely differ in their order usually possess interesting integral properties. One of the most important of these is known as the orthogonal property, and has undergone a renascence in recent years. We can illustrate by Legendre's function $P_n(x)$, which satisfies the equation

$$(1 - x^2)P_n'' - 2xP_n' + n(n + 1)P_n = 0.$$

Similarly for a different order m we have

$$(1 - x^2)P_m'' - 2xP_m' + m(m + 1)P_m = 0.$$

On multiplying the first by P_m and the second by P_n, we reach

$$\{m(m + 1) - n(n + 1)\}P_mP_n$$
$$= 2x(P_m'P_n - P_mP_n') - (1 - x^2)(P_m''P_n - P_mP_n'')$$
$$= \frac{d}{dx}\{(1 - x^2)(P_mP_n' - P_m'P_n)\}.$$

The right side suggests that we integrate between the limits +1 and −1, whereupon we reach

$$\int_{-1}^{1} P_mP_n\,dx = 0.$$

This is the orthogonal property which connects two Legendre functions of different orders. An even simpler orthogonal relation, with which the reader has long been familiar, comes from the oscillation equation

$$\frac{d^2y}{dx^2} + n^2y = 0,$$

viz. $$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0, \quad m,\ n \text{ integers}.$$

EXERCISES

1. Prove that the Laguerre function, mentioned in 11, 7, Ex. 2, has the orthogonal property

$$\int_0^\infty e^{-x} L_n L_m \, dx = 0.$$

2. The Tschebyscheff function T_n of order n satisfies the equation

$$(1 - x^2)y'' - xy' + n^2y = 0.$$

Prove that it has the orthogonal property

$$\int_{-1}^{1} T_n T_m (1 - x^2)^{-\frac{1}{2}} dx = 0.$$

3. The Hermite function H_n of order n obeys the differential relation

$$H_n' = 2nH_{n-1},$$

and has the recurrence formula

$$H_{n+1} - 2xH_n + 2nH_{n-1} = 0.$$

By differentiating the latter, show that $H_n(x)$ satisfies the equation

$$y'' - 2xy' + 2ny = 0,$$

and deduce the orthogonal property

$$\int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx = 0.$$

11, 9. *Solution in Series.*

When other methods of integration fail, and the coefficients in the differential equation are polynomials in x, it is sometimes possible to derive a solution in a series of ascending powers of x. The technique of the method is simple enough. One merely assumes that

$$y = x^r(a_0 + a_1x + a_2x^2 + \ldots).$$

The various differential coefficients can be derived from this and substituted in the given equation, the result being rearranged in ascending powers of x. As the equation is supposed to be identically satisfied, the coefficients of the various powers of x must all be zero. This gives a sequence of relations between the a's, known as the recurrence relations, which serve to determine the higher a's in terms of the first one or two. The coefficient of the lowest power of x gives what is known as "the indicial equation" which furnishes the permissible values of r.

We can exemplify the method by solving the equation

$$x\frac{d^2y}{dx^2} + y = 0.$$

In spite of its innocuous appearance it happens to be a form of Bessel's equation. We have

$$\begin{aligned} xy'' &= a_0r(r-1)x^{r-1} + a_1(r+1)rx^r + a_2(r+2)(r+1)x^{r+1} + \dots, \\ y &= \qquad\qquad\qquad\quad a_0x^r + \qquad\qquad\qquad a_1x^{r+1} + \dots . \end{aligned}$$

On summation, the first term gives $a_0r(r-1) = 0$ as the indicial equation. The coefficient a_0 cannot be zero (the series must start somewhere); alternatively, we have $r = 0, 1$.

The zero value is useless; if we apply it to the coefficient of x^0 we again get a_0 as zero. We accordingly try $r = 1$, and we have in succession

$$1 \cdot 2a_1 = -a_0, \qquad a_1 = -\frac{a_0}{1 \cdot 2}.$$

$$2 \cdot 3a_2 = -a_1, \qquad a_2 = -\frac{a_1}{2 \cdot 3} = +\frac{a_0}{1 \cdot 2 \cdot 2 \cdot 3}.$$

$$3 \cdot 4a_3 = -a_2, \qquad a_3 = -\frac{a_2}{3 \cdot 4} = -\frac{a_0}{1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 4}.$$

$$\dots = \dots, \qquad\qquad \dots = \dots .$$

The solution can be written

$$y = a_0\left\{x - \frac{2x^2}{(2!)^2} + \frac{3x^3}{(3!)^2} - \frac{4x^4}{(4!)^2} + \dots\right\},$$

where the constant a_0 remains arbitrary.

The result flatters the method more than somewhat. The law of formation of the coefficients is particularly easy to see, and the series is convergent for all values of x. This is by no means always the case, nor is it true in general that such a series solution necessarily exists. A trained eye can tell from an examination of the coefficients in the differential equation what to expect in any particular case; but a full understanding of the matter really requires a knowledge of the complex variable. It will be noticed from the above analysis that not every root of the indicial equation necessarily leads to a solution of the required type. It will be stated here without proof that if the two roots of the indicial equation are the same, not more than one solution of the required type exists, and the same is usually (but not always) true if the two roots differ by an integer.

EXERCISES

1. Weber's equation can be written in the form

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + ny = 0.$$

Show that c is necessarily zero and the solution is two series, multiplied by a_0 and a_1 respectively. As these are independent, two series solutions are forthcoming, viz.

$$y_1 = 1 - \frac{n}{2!}x^2 + \frac{n(n-2)}{4!}x^4 - \ldots,$$

$$y_2 = x - \frac{n-1}{3!}x^3 + \frac{(n-1)(n-3)}{5!}x^5 - \ldots.$$

2. The following equation occurs in the theory of ionic diffusion:

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + c(a^2 - r^2)\varphi = 0.$$

Prove that the formal method gives a series, in ascending even powers of r, the first few terms being

$$\varphi = 1 - \frac{ca^2}{4}r^2 + \frac{1}{16}\left(c + \frac{c^2a^4}{4}\right)r^4 + \ldots$$

See J. J. Thomson, *Conduction of Electricity through Gases*.

3. The following equation occurs in the difficult dynamical problem of the rolling disc; see Routh, *Advanced Rigid Dynamics*:

$$(1 - x^2)y'' - 2xy' - \beta y = 0.$$

It is known to be soluble in terms of Legendre functions of complex order. Prove that there are two series solutions, the one in even powers and the other in odd powers.

4. A deep, narrow beam is uniformly loaded along its upper edge. The condition of stability is derived from an equation of the type

$$\frac{d^2\theta}{dx^2} + (a + bx^4)\theta = 0.$$

See Prescott, *Applied Elasticity*, p. 522.

Choosing a new variable defined by $t = x^2$, establish the formal existence of a series solution in ascending powers of x^2 and find the first few terms.

Miscellaneous Exercises

1. A projectile of mass m is fired horizontally with velocity u into sand whose resistance is $av + bv^2$, where v is the velocity at any time t. Prove that the total penetration s is given by

$$\frac{bs}{m} = \log\left(1 + \frac{b}{a}u\right),$$

and that the time of transit is given by

$$\exp\left(\frac{at}{m}\right) = \frac{u(a + bv)}{v(a + bu)}.$$

Deduce from a consideration of the time that the law of resistance cannot be wholly valid. Deduce also from first principles that a law of resistance $av - bv^2$ is likewise invalid unless the range of values of v be restricted.

2. A body of mass m is fired vertically upward from the ground with velocity u. The atmospheric resistance is $av + bv^2$ when v is the velocity. Prove that the body will rise to a height s given by

$$\frac{bs}{m} = \frac{1}{(\alpha - \beta)}\log\frac{(1 - u/\alpha)^{\alpha}}{(1 - u/\beta)^{\beta}},$$

where α, β are the roots of the quadratic $bv^2 + av + mg = 0$. Prove also that the time of flight is given by

$$\tan\frac{dbt}{m} = \frac{2dbu}{au + 2mg},$$

where

$$d^2 = \frac{mg}{b} - \left(\frac{a}{2b}\right)^2.$$

3. A uniform straight rod of length λ and mass m has its lower end smoothly hinged. It starts from rest in the all but vertical position. Prove that the time from the horizontal to the downward vertical is given by

$$t\left(\frac{3g}{\lambda}\right)^{\frac{1}{2}} = 2\sinh^{-1}1 = 1{\cdot}76.$$

4. It is found that over a certain range of velocities the resistance per unit mass has the form $av - bv^{1\cdot6}$. Prove that, in the absence of other forces, the velocity at any time t is given by a relation of the form

$$v^{-0\cdot6} = A\exp(0{\cdot}6at) + \frac{b}{a}.$$

5. A moving body is opposed by a force proportional to the displacement and by a resistance proportional to the square of the velocity (quadratic damping). Prove that the equation of motion has the form

$$\frac{d^2x}{dt^2} + b\left(\frac{dx}{dt}\right)^2 + cx = 0,$$

on the assumption that the displacement is increasing. Verify that the substitution $(dx/dt)^2 = y$ makes the equation linear and deduce that the velocity v is connected with the displacement by the relation

$$v^2 = ae^{-2bx} - \frac{cx}{b} + \frac{c}{2b^2}.$$

Note that the equation of motion cannot hold in both directions; for if x be reversed, the middle term retains its positive sign.

6. If M, N are two homogeneous functions of the same degree, prove by the use of Euler's theorem on homogeneous functions that the homogeneous equation $M\,dx + N\,dy = 0$ has the integrating factor $(Mx + Ny)^{-1}$, which makes it exact. Apply the method to

$$x^3\,dx + y(3x^2 + 2y^2)\,dy = 0$$

to show that it has the solution $x^2 + 2y^2 = c\sqrt{(x^2 + y^2)}$.

7. A variable line cuts a curve in two points A, B. The tangents at these points intersect in T. The line joining T to C, the mid-point of AB, has a constant direction. It is required to find the form of the curve. Keeping A fixed, take A as the origin and AT as the x axis. For variable B, let TC make the constant angle $\theta = \tan^{-1} m$ with AT. Deduce that the differential equation of the curve is

$$2my = (mx - y)\frac{dy}{dx},$$

leading to the parabola $cy = (y + mx)^2$. The problem occurs in the theory of the suspension bridge.

8. Experiment shows that a parachute of area A square feet moving with velocity v ft./sec. experiences atmospheric resistance $Av^2/800$ pounds. Solve the equation of motion for a parachutist whose weight in harness is 10 stone, the area of his parachute being 600 square feet. Deduce his terminal velocity and verify from first principles. What difference does it make if the parachute takes 3 seconds to open and cushion out, taking those 3 seconds as the man's time of free falling before the resistance begins to operate?

9. If AB is part of a uniform catenary, the condition of equilibrium demands that the intersection of the tangents at A and B should be vertically below the centre of gravity. This property suffices to determine the form of the curve. Take A as the vertex and the origin. A consideration of $\bar{x}$ leads to

$$s\left(x - \frac{y}{p}\right) = \int x\,ds.$$

Differentiation with respect to x leads to $s = cp$, which is the usual intrinsic equation $s = c\tan\psi$.

10. The intercept, between the co-ordinate axes, on any tangent to a certain curve is of constant length. It is required to find the form of the curve. Note that if the intercept c is in the fourth quadrant we have geometrically

$$y = x\tan\psi - c\sin\psi, \text{ where } \tan\psi = dy/dx.$$

[The four-cusped hypocycloid $x^{2/3} + y^{2/3} = c^{2/3}$.]

11. A rotor of inertia I is driven by a couple which varies with the time and has the form $A \sin \omega t$. Prove that there are three possibilities to the motion: (i) the rotor may move continuously in one direction with varying speed, (ii) it may swing to and fro through an angle $A/I\omega^2$ about some mean position, (iii) it may move to and fro whilst maintaining on the whole an ever-increasing displacement in one direction.

12. Find the differential equation of the families

$$\text{(i) } y = Ab^x, \quad \text{(ii) } y = Ax^b.$$

13. Find the orthogonal trajectories of the family

$$\frac{x^2}{a^2} + \frac{y^2}{\lambda^2} = 1,$$

where λ is the parameter.

14. A ship A pursues a straight course with constant speed u. A second ship B pursues a curved course with constant speed v and its motion is always in the direction AB. Find the equation of B's path parametrically.

15. If y_1, y_2 are solutions of $y'' + y'f(x) + yg(x) = 0$, prove by eliminating $g(x)$ that $y_1'y_2 - y_1y_2' = A \exp(-\int f dx)$. Hence, if one solution is known, another can be found. Illustrate by using the fact that $y_1 = x$ is a solution of

$$(1 - x^2)y'' - xy' + y = 0.$$

16. If y_1 is a solution of $y'' + y'f(x) + yg(x) = 0$, prove that $y_2 = uy_1$ is also a solution provided that

$$y_1 \frac{dw}{dx} + w\left(y_1 f + 2\frac{dy_1}{dx}\right) = 0, \quad w = \frac{du}{dx}.$$

Hence find a second solution of

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0,$$

given that $y_1 = x$ is a solution.

17. Extend the method of the previous question to the equation

$$y'' + y'f + yg = h$$

by showing that a solution can be found if a solution of the reduced equation is known. Hence derive a solution of

$$xy'' + (1 - x)y' + y = (3x + 1)e^x,$$

given that $y = (1 - x)$ is a solution of the reduced equation.

18. A mass m lies on a rough horizontal table and is attached to one end of a string of length λ. The other end of the string is very slowly drawn along the table edge. Find the path of the mass. [The curve is known as a tractrix and is an involute of the catenary. It finds practical application in Schiele's bearing.]

19. Since the result of differentiating a differential equation is to produce an equation whose order is higher by unity, it must occasionally happen that

The order of an equation can be reduced by unity if we integrate each term by parts. For example, the equation

$$y'' - y'\cot x + y \operatorname{cosec}^2 x = \cos x$$

integrates directly to $\quad y' - y\cot x = \sin x + a,$

and the solution can be completed by other means.

Integrate $x^2y'' + 3xy' + y = 4x$ once in this manner. Prove that the process can be repeated and leads to $xy = x^2 + a\log x + b$. If the equation has the form $Py'' + Qy' + Ry = S$, prove that the method is applicable if $R - Q' + P'' \equiv 0$. Check by the above examples and by the equation $(x^2 - 1)y'' + 5xy' + 3y = 4x$.

Extend the principle to the nth order equation,

$$X_0\frac{d^ny}{dx^n} + X_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + X_ny = f(x),$$

and prove that the method is certainly applicable if each coefficient is a polynomial of degree lower than the order $(n - r)$. Test by the equation

$$(x^2 - x)y''' + xy'' - y' = 5x.$$

20. Prove that the equation $y'' + Py' + Qy'^2 = 0$ can be integrated if P, Q are both functions of y or both functions of x. Illustrate by

$$y'' + y'\sin 2y + y'^2\sec y = 0.$$

21. A reduced linear equation of the second order has the two independent solutions $y_1 = \sin x$ and $y_2 = x\cos x$. Prove that the equation is

$$y''(\sin x\cos x - x) + 2y'\sin^2 x = y(\sin x\cos x + x).$$

Conversely, solve this equation after verifying that $\sin x$ is a solution.

22. If P, Q are functions of x, prove that the equation

$$\frac{dy}{dx} + Pe^y = Q$$

can be reduced to the linear form.

23. Subsoil water drains into a circular sump of radius r. The water is pumped out as fast as it accumulates, so that the water-level is at constant height h above sump-bottom. At horizontal distance x from the centre-line, the water-level in the subsoil is at height y above the sump-bottom. An imaginary cylindrical surface surrounding the sump would thus have a surface area $2\pi xy$. It is assumed that water flows horizontally inwards across this surface with velocity proportional to the water-surface gradient; or, $v = k\,dy/dx$. As water is incompressible and the sump-level is constant, the inward flow per second across any such imaginary cylindrical surface is constant, say A. Write down the equation of flow, integrate it, and determine the constant from the conditions in the sump.

How does your result show that the argument is fallacious?

$[\pi k(y^2 - h^2) = A\log(x/r).]$

24. According to Exercise 1, 8 (4) the differential equation to the catenary $y = c\cosh(x/c)$ is $y\sinh^{-1}y' = x\sqrt{(1 + y'^2)}$.

Integrate this equation.

25. Show that the substitution $y = \exp\int w\,dx$, or $y'/y = w$, when applied to the second order linear equation with constant coefficients $ay + by' + cy'' = 0$, reduces the equation to the first order with the variables separable.

26. Prove that the generalized Clairaut equation $y = xf(p) + F(p)$, if differentiated with respect to x, takes the linear form

$$\{f(p) - p\}\frac{dx}{dp} + xf'(p) + F'(p) = 0.$$

Hence, prove that the equation $x + yy' = cy'^2$ leads to $x \coth u = a + cu$, where $p = \sinh u$.

27. Prove that in Riccati's equation

$$x\frac{dy}{dx} - ay + by^2 = cx^n$$

the variables can be separated by the substitution

$$y = x^a v \text{ if } n = 2a.$$

Hence, prove that the equation

$$x\frac{dy}{dx} - 2y - y^2 = x^4$$

has the solution $$y = x^2 \tan(c + \tfrac{1}{2}x^2).$$

28. The stability of a thin beam provides the equations

$$\frac{d^2z}{dx^2} = a\theta, \quad \frac{d^3\theta}{dx^3} - b\frac{d\theta}{dx} = c\frac{dz}{dx} + d,$$

where the constants are all positive. Find a simple solution that makes θ zero when x is 0 or λ, and state the conditions.

29. The stability of a rod under thrust and torsion depends on the simultaneous equations

$$\frac{d^2y}{dx^2} - a\frac{dz}{dx} + by = 0,$$

$$\frac{d^2z}{dx^2} + a\frac{dy}{dx} + bz = 0,$$

where a, b are positive. By taking a new dependent variable $w = y + iz$, prove that solutions are given by $w = \exp i\lambda x$, where λ is real but may be positive or negative. Verify independently that there are solutions of the type $y = \cos\lambda x$, $z = \sin\lambda x$.

30. For a shaft with end-thrust and torque, we have

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} - m\omega^2 y = T\frac{d^3z}{dx^3},$$

$$EI\frac{d^4z}{dx^4} + P\frac{d^2z}{dx^2} - m\omega^2 z = -T\frac{d^3y}{dx^3}.$$

Solve as in the previous example.

31. The following occurs in the theory of beams of variable section

$$(1 + ax)^2 \frac{d^2y}{dx^2} + a(1 + ax) \frac{dy}{dx} + by = 0.$$

Solve by taking a new independent variable.

32. The theory of thick spherical shells leads to the equation

$$\frac{d^2p}{dr^2} + \frac{4}{r}\frac{dp}{dr} = 0.$$

Prove that it has the solution $p = b - \frac{a}{r^3}$.

33. Solve the equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 V = 0$$

by first proving that the equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) U = 0$$

has the solution $U = a + b \log r$.

34. The second order equation with variable coefficients can be solved when the operator can be factorized. Illustrate by verifying that the equation

$$2x(x + 2)y'' + (x^2 + 3x - 2)y' - 3(x - 1)y = 0$$

can be written $$[2xD + (x - 1)][(x + 2)D - 3]y = 0.$$

A solution can then be obtained by writing

$$\{(x + 2)D - 3\} y = z,$$

$$\{2xD + (x - 1)\} z = 0.$$

35. The equation $(2x - 1)^3 y''' + (2x - 1)y' = 2y$

can be treated as homogeneous linear, and leads to

$$y = v(a + bv^{\alpha} + cv^{-\alpha}), \quad v = 2x - 1, \quad \alpha = \tfrac{1}{2}\sqrt{3}.$$

36. The deflection of a circular plate under uniform load is given by

$$\frac{1}{r}\frac{d}{dr}\left[r \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r \frac{dw}{dr}\right)\right\}\right] = c.$$

Prove that this is equivalent to

$$w^{iv} + \frac{2}{r} w''' - \frac{1}{r^2} w'' + \frac{1}{r^3} w' = c.$$

Solve on the assumption that when r is zero, w is finite and dw/dr zero.

37. If the transverse vibrations of a beam are forced, we have

$$\frac{\partial^4 y}{\partial x^4} + k^2 \frac{\partial^2 y}{\partial t^2} = a \sin \omega t.$$

Find a particular integral by assuming that $y = X \sin \omega t$.

38. The equation for a whirling shaft subject to end thrust P is

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} - m\omega^2 y = 0.$$

Find the smallest value of ω for stability if the end conditions are

$$x = \pm\tfrac{1}{2}\lambda, \quad y = 0 = y''.$$

39. In the discussion of the transverse vibrations of a bar it was assumed that the couple $F\,dx$, due to shear, equilibrated with the excess bending moment dM. More accurately we might say that their difference changed the angular momentum of the element AB. The inertia of the element is $I\rho\,dx$, where ρ is the density. Its angular acceleration is $\partial^2\psi/\partial t^2$, where $\psi = \partial y/\partial x$. Deduce that the modified equation is

$$\frac{\partial^2 y}{\partial t^2} - k^2\frac{\partial^4 y}{\partial x^2\,\partial t^2} + \frac{Ek^2}{\rho}\frac{\partial^4 y}{\partial x^4} = 0,$$

where k is the radius of gyration of the cross-section. Note that the general method of substituting $y = TX$ fails. What other method is available?

40. A railway transition curve is to be laid out on the principle that the curvature varies uniformly as we proceed along the track. This gives the equation $d\rho^{-1}/ds =$ constant, and (as a matter of convenience) this constant can be taken as $2b^2$. Taking the straight part of the track as x axis and the departure from the straight as origin (where the curvature is zero), prove that any point on the curve can be expressed as

$$x = \int_0 \cos(bs)^2\,ds, \quad y = \int_0 \sin(bs)^2\,ds.$$

41. If the isoclinals are a family of parallel straight lines, prove that the equation can be solved by separation of the variables.

42. The lower edge of a vertical sheet of absorbent material is placed in contact with liquid. Let x be the rise after time t. Assume that m is the mass of liquid per unit surface area of the sheet; p is the capillary lift per unit length at the top of the stain. Show that this leads to an equation of the form

$$p - mgx = m\frac{d}{dt}\left(x\frac{dx}{dt}\right).$$

Find a particular integral and interpret it. Find also a first integral of the equation by the use of an integrating factor.

43. For a thin hemispherical manhole end in a high-pressure welded drum, it can be shown that

$$\frac{d}{d\theta}(F\sin\theta) = (a - T)\sin\theta,$$

$$\frac{d}{d\theta}(T\sin\theta) = F\sin\theta + b\cos\theta.$$

If a, b are constants, whilst F, T are functions of θ, prove that these lead to

$$F = \tfrac{1}{2}(a - b)\theta + p + q\cot\theta,$$

where p, q are arbitrary.

44. In certain circumstances, the conditions for the stability of a beam depend on an equation of the type

$$\frac{d^2\theta}{dx^2} + c\theta x^4 = 0.$$

Derive a series solution in ascending powers of x^6.

Hints and Answers

1, 8.

5. Solve for $(a^2 + \lambda)^{-1}$ and $(b^2 + \lambda)^{-1}$, then invert.
8. Begin by multiplying by $\cos x$.
10. Use logarithmic differentiation on $(x - a)(y - b) = c$.
14. ae^c is effectively a single constant.

2, 5.

5. Use $x \equiv (a + x) - a$.
10. $\exp(-\sin x)$ ranges between e and $1/e$.
17. Take a as the initial radius and x as the variable radius. Surface area is proportional to x^2; strength of solution is proportional to $a^3 - x^3$; rate of solution is proportional to $-d(x^3)/dt$.
18. After solving the equation, use the given values to eliminate the constants. This gives $10[2^{n-1} - 1] = 3[(10/3)^{n-1} - 1]$. Trial of $n = 2, 3, 4$ shows that 3 gives the best fit.
21. Dimensions are cm.$^{-1}$.
22. Taken from a research. Use partial fractions. Long rather than difficult.
25. $t = x + y$ leads to $x + y = \tan(x + \frac{1}{4}\pi - \frac{1}{2})$.
27. Put $t = au + bv$ and discard u.
28. As $|x| \leqq a$, $|y| \leqq b$, the family are ellipses in a rectangle.

2, 8.

(i) $2xyy' = y^2 - x^2$; (ii) centre displaced; (iii) $\dfrac{dP}{dV} + \dfrac{P}{V} = 0$; (iv) radius altered; (v) $\dfrac{dy}{dx} = \dfrac{y}{2x}$.

2, 9.

2. If $\dot{x} = ax + by + c$, $\dot{y} = px + qy + r$, then $a + q = 0$.

2, 15.

3. $J = x$, $c = x^2y(2 - y)$.
4. $x^2 \cos y - y^3 \tan x + f(y)$.

3, 3.

5. If $y = Ae^{\alpha x} + Be^{\beta x}$, then $A + B = 0$. The conditions $\alpha = \beta$ and $\alpha \neq \beta$ both fail.

3, 6.

4. Condition depends on $2x = 1 - e^{-2x}$. Consider the graph of each side.
8. $D = \dfrac{\omega}{\sqrt{2}}(\pm 1 \pm i)$, $t = \dfrac{x\omega}{\sqrt{2}}$, $y = (ae^t + be^{-t}) \cos t + (ce^t + de^{-t}) \sin t$.

3, 9.

6. The asymptote is crossed once, when x is negative. The axis OX is crossed, or not, according as the minimum value is negative or positive.

4, 1.

2. The family has the Cartesian equation $x^2 + y^2 - 2x(r^2 - c^2)^{\frac{1}{2}} = c^2$. The orthogonal family has the differential equation

$$y' - \frac{c^2y'}{y^2} + \frac{2xy - x^2y'}{y^2} = 0.$$

4. The family is self-orthogonal.

4, 2.

6. Use the two forms $\dfrac{dy}{c} = \dfrac{p\,dp}{(1+p^2)^{\frac{3}{2}}}$, $\dfrac{dx}{c} = \dfrac{dp}{(1+p^2)^{\frac{3}{2}}} = d\left\{\dfrac{p}{(1+p^2)^{\frac{1}{2}}}\right\}$.

7. A circle with centre on OX.
8. Harmonic motion about a new origin $x = a/n^2$.
10. It reaches infinity with positive velocity if $mhu^2 > 2\mu$.
12. Write $3p'/p = 2p''/p'$.

4, 3.

3. Note that the equation is completely solved by the substitution

$$2y - x + 1 = t^2.$$

4, 4.

4. The four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.

5, 4.

4. Calculate $2\ddot{y} - \ddot{x}$.

5. Initially there is no displacement and no velocity; hence the acceleration components are a, 0.

6. The quadratic $(D^2 + \omega^2)^2 + c^2D^2 = 0$ shows that D^2 cannot be positive, but is real. There are two values of n with opposite signs.

9. Elimination gives four values for D, two of which are zero. The first equation then shows that θ does not contain the secular term.

6, 10·3.

5. $x \sin x$ is the product of two skew functions and is therefore symmetrical.
6. A sketch shows the analysis to be even cosines.
7. A sketch shows that $y - \frac{1}{2}\pi$ is expressible in even sines.
9. Use the function $x(\pi - x)$ and afterwards change the period.

$$x(\lambda - x) = \frac{8\lambda^2}{\pi^3}\left\{\sin\frac{\pi x}{\lambda} + \frac{1}{3^3}\sin\frac{3\pi x}{\lambda} + \cdots\right\}.$$

14. The range can be completed in an infinite variety of ways.
15. $f(x) - f(-x)$; $f(x) \equiv \frac{1}{2}[\{f(x) + f(-x)\} + \{f(x) - f(-x)\}]$; $f(x) + f(x+\pi)$.
16. Depends on expressing $\cos^n\theta$ in multiple cosines; see any work on higher trigonometry.

19. If the Fourier series be squared we have eight types of term. Most of these give zero on integration. The mean value of y^2 is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} y^2\,dx = [a_0^2 + \tfrac{1}{2}\Sigma(a_n^2 + b_n^2)] > a_0^2.$$

21. Similar to 19.

23. Expressible in odd sines.

24. Expressible in odd sines. Take $y = \frac{1}{2}\pi - x$, $0 < x < \frac{1}{2}\pi$ and afterwards change the scales.

$$y = \frac{8}{\pi^2}\left[(\tfrac{1}{2}\pi - 1)\sin\frac{\pi x}{2} + \tfrac{1}{3}(\tfrac{1}{2}\pi + \tfrac{1}{3})\sin\frac{3\pi x}{2} + \tfrac{1}{5}(\tfrac{1}{2}\pi - \tfrac{1}{5})\sin\frac{5\pi x}{2} \dots\right].$$

7, 11.

7. (i) Put $az + cx = at$; (ii) $2az + cx^2 = 2at$; (iii) divide by z.

9. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is a solution provided $a + b = 0$.

10. Needs $2a + 2b = k$.

11. The assumption $V = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ gives $a + c = 0 = b + d$. The given result comes from $a + d = 0$; see also the remarks in 2, 9·1.

12. m, n may have to be integers, or the halves of odd integers.

$$m^2 + n^2 = a^2p^2/\pi^2$$

and is at least $\frac{1}{2}$.

15. (i) A free end has no couple and $d\theta/dx$ is zero. (ii) A fixed end has no movement and θ is zero. (iii) At a loaded end the couple has a given value.

19. Based on No. 12.

20. Put $\xi = Ue^{-nt}$ and the equation becomes ordinary.

22. Use $\partial r/\partial x = x/r$, and similar expressions, to give

$$\frac{\partial V}{\partial x} = -\frac{x}{r^3},\quad \frac{\partial^2 V}{\partial x^2} = -\frac{1}{r^3} + \frac{3x^2}{r^5}.$$

8, 8.

3. The equation relates to the family $(x - c)(y - x + c) = 4$.

4. The equation is soluble by $t = x - y$. It comes from $(y - x)(x - c) = 4$.

8. Derives from $(x^2 - y^2)(y - 1) = c$.

9. The isoclinals are $y = \log(x - c)$ and the inflection locus is $x = 2\cosh y$.

9, 3.

3. The equation is soluble; $y = 6e^x - 5 - 2x - x^2$.

10, 2.

1. $dx + dy + dz = 0 = \dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z}$.

3. Calculate $d(x + y + z)$, $d(x + y - 2z)$, $d(y + z - 2x)$.

4. The four lines are $x = \pm y = \pm z$.

5. Use $d(y - z)$.

8. The initial motion can be deduced without solution. The point finally goes to infinity in the third quadrant.

9. Calculate $d(x - z)$ and $d(2x + z)$.

11. The calculation of the differential $d(yz)$ leads to $yz + x^2 = a$. Treat $d(xz)$ similarly. See also the next section, 10, 3.

10, 3.

4. Take the point as origin. The direction $x : y : z$ is then perpendicular to $p : q : -1$.

5. Use $x^2 + y^2 = c^2$ to eliminate y. This gives $\dfrac{x}{c} = \sin\left(\dfrac{z}{a} + b\right)$ so that x (and y) are unchanged when z changes by $2\pi a$.

6. $z = (x - y)^2$.

7. Two integrals are $x^2 - y^2 - z^2 = a$, $z^2 - 2xy = b$.

9. $z = x^2 + y^2 - a$.

10. A family of planes through OZ; a family of spheres through the origin, with centres on OX. If $f\left(\dfrac{y}{x}\right) = a\left(\dfrac{y}{x}\right) + b$, then

$$xf\left(\frac{y}{x}\right) = ay + bx = x^2 + y^2 + z^2.$$

10, 4.

1. Two integrals of the subsidiaries are $3x - 2y = a$, $x - z = b$, whence $4b^2 + (3b - a)^2 = 16$.

2. $yz = xf(x^2 + y^2 + z^2)$.

10, 8.

3. Regroup as $(x\,dy + y\,dx) +$ etc.

4. $xy^2 + x^2(1 + z) = a$.

5. $z = y(x - a)$.

11. (i) $x^2 - xy + y^2 = cz$, (ii) $y(x + z) = c(y + z)$.

11, 2.

1. A first integral is $c + \log v' = \frac{1}{2}x^2 - 3x - 2\log(x - 1)$.

2. $y = a\cos x + be^{-x}$.

11, 5.

2. The equation shows that v and v'' have opposite signs. As v'' must change sign in the range, so does v.

Miscellaneous Exercises.

1. (i) The body requires infinite time to come to rest; (ii) certain velocities give negative resistance.

8. About 14 feet per second.

12. (i) $yy'' = y'^2$. (ii) $\dfrac{y''}{y'} - \dfrac{y'}{y} + \dfrac{1}{x} = 0$.

13. $x^2 + y^2 = 2a^2 \log x + b$.

14. $y^{u/v} = c \cot \frac{1}{2}\psi$.

16. $w = \dfrac{du}{dx} = \dfrac{ae^{x^2}}{x^2}$.

18. $y = c\sin\psi$, whence x can be found in terms of ψ.

20. A first integral is $p(\sec y + \tan y) + 2\int(\sin y + \sin^2 y)\,dy = c$.

22. Put $e^{-y} = u$.

23. The result makes y increase indefinitely with x, whereas y cannot exceed the depth of the sump.

24. Choose a new variable defined by $p = \sinh u$; this leads to $x = cu$.

34. $\dfrac{d}{dx}\left\{\dfrac{y}{(x+2)^3}\right\} = \dfrac{c(xe^{-x})^{\frac{1}{2}}}{(x+2)^4}$.

38. See Case, *Strength of Materials*, p. 506.

39. See Lamb, *Dynamical Theory of Sound*, p. 125.

41. The family is $c = f(ax + by) = \dfrac{dy}{dx}$. Put $ax + by = t$.

42. The final position is $x = p/mg$.

$$\left(\frac{dx}{dt}\right)^2 = \frac{p}{m} - \frac{2gx}{3} - \frac{1}{3(gx)^2}\left(\frac{p}{m}\right)^3.$$

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